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ANALYSIS OF GAS FLOW SYSTEMS FOR DYNAMIC CONTROL PURPOSES

By

W. K. McGregor, Jr. R. W. Messick

D. W. Russell L. F. Burns

ETF, ARO, Inc.

April 1956

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FOREWORD

An investigation was initiated in 1950 of the problems involved in the design of an automatic pressure control system for the Engine Test Facility and Ram-Jet Addition, Arnold Engineering Development Center. An analysis of the plant was made, and the performance of all components of the control system was specified.

Individual responsibilities for different phases of the investigation are as follows:

G. V. Schwent - determined the basic schemes of analysis and directed the study.

D. W. Russell - supervised work on distributed parameter equations and completed the lumped parameter analysis.

W. K. McGregor and L. F. Burns - derived distributed parameter equations and determined the frequency response solutions.

Roger Messick - obtained the indicial solution to the distributed parameter equations.

C. T. Coffee - contributed suggestions and directed the mathematical calculations.

The report was compiled by Mr. McGregor.

CONTENTS

ILLUSTRATIONS

ABSTRACT

Methods are developed for the prediction of frequency and indicial response of gas flow systems on both a lumped parameter and a distributed parameter basis. The laws of physics which apply to gas flow are derived, and from them the differential equations which govern the system response are developed. The methods of operational calculus are used to solve these equations for the case of sinusoidal and indicial forcings. The various implications of these solutions with their bearing on control of pressure variables are discussed. Examples are presented in the appendices to add clarification to the method and its implications. The report correlates existing knowledge of gas flow systems as a basis for formulating an inclusive reference to be used in the design of gas flow systems having desirable dynamics and in the design of control systems.

NOMENCLATURE

The gravitational, dimensional system will be used throughout. The numerical unit system of Appendix 3 will use the British units; pounds force, feet, second, degrees Rankine.

A	Area of valve restriction
A_0	Duct cross-sectional area
$A^*, B^*, C^*, D^*, E^*, F^*$ –	Boundary value functions of s or $j\omega$
C	Lumped pneumatic capacitance
G	Magnitude ratio (or gain)
K	Arbitrary constant
L	Total length of ducting
M	Mach Number
P	Absolute pressure
Q	A defined function of s, $Q = \left(\frac{r + hs}{hs}\right)^{1/2}$
R	Ideal gas constant; In Appendix 3, R = 53.3 ft/°R
R^*	Residue
U	Velocity of moving fluid in a duct
V	Volume
W	Weight flow of gas
W^*	Weight of gas
Z	Terminal pneumatic impedance
a	Incremental change (or small perturbation) in valve area from the steady state value, A
b	Used in definition of e
c	Distributed pneumatic capacitance

f	Function of one or more variables
h	Distributed pneumatic inertance
j	The quantity $\sqrt{-1}$
k	Assigned constant denoted by subscript; integer without subscript
n	Integer
p	Incremental change (or small perturbation) in pressure from the steady state value, P
r	Distributed pneumatic resistance
s	The Laplace complex operator
t	Time
u	Velocity of pressure wave propagation (velocity of sound)
w	Incremental change (or small perturbation) in weight flow from the steady value, W
x	Distance
y	Distributed pneumatic admittance
z	Distributed pneumatic impedance
α, β	The real numbers defined by $\sqrt{zy} = \alpha + j\beta$
γ	Ratio of specific heats ($\gamma = 1.4$ for diatomic gas)
δ	A defined quantity
e	The quantity $\lim\limits_{x \to 0} (1 + x)^{1/x}$, or 2.178 . . .
ζ	The dimensionless damping ratio of a second order differential equation
η	Integer
Θ	Absolute temperature
θ	Incremental change (or small perturbation) in absolute temperature about the steady state value Θ
λ	Wave length of a sinusoidal pressure wave

π	The factor 3.1416 . . .
ρ	Gas density
σ	The quantity defined by $e^{2\sigma\pi} = \dfrac{Z - \sqrt{\frac{h}{c}} \quad Q}{Z + \sqrt{\frac{h}{c}} \quad Q}$
τ	Time constant
ϕ	Phase angle
ω	Angular frequency of oscillation

Subscripts:

Numerical subscripts 1, 2, 3 etc., and alphabetical subscripts a, b, c, etc., are used principally to distinguish between like expressions such as f, k, and w.

Alphabetical subscripts i, o, and x are specifically used to designate stations --i being used to designate inlet and o to designate outlet.

Dot Notation is used to express derivative with respect to time, e.g., $\frac{dP}{dt} = \dot{P}$.

INTRODUCTION

Many industrial organizations have undertaken extensive analyses of their processes in order to determine more exactly their control requirements. One of these processes concerns the control of pressure and flow of gases through systems of ducting, referred to as gas flow systems in this paper. An example of such a system is a wind tunnel in which airframe models or power plants are tested under accurately simulated Mach number and altitude conditions.

Fundamental to the design of a physical plant and its control system is knowledge of the behavior of variables in the system in response to various disturbances. An understanding of the change in behavior which results from varying the different factors and properties of the physical design is also required. Methods of acquiring this information for gas flow systems are developed in subsequent sections.

This report assembles the applicable physical relations and formulates the response of pressure in gas flow systems to expected disturbances. There does not exist a treatment of this problem of sufficient scope and clarity to be useful to the control designer. To be sure, the laws of physics which apply are not new and may be found in various textbooks on thermodynamics, aerodynamics, acoustics and fluid flow. However, the objective of this paper is the use of these laws and their formulation into a useful tool. The principal applications of previous investigations of gas flow dynamics have been concerned with such things as pressure measurement instrumentation and acoustic muffler design. The mathematical similarity between the laws of gas flow through ducts and the laws of electricity affords an analogy that has been used advantageously by previous investigators (Ref. 1).

The principal difference between this work and existing ones on the same general subject is that here the gas flow rather than pressure is considered as a forcing function. This is necessary in an analysis for control system design because both the major disturbance and the manipulated variable are gas flows.

The methods of operational calculus are used in this report because of the uniformity of expression of the different concepts and because of the almost universal use of the Laplace Transform and transfer function notation in the synthesis of control systems. The frequency response and indicial response are used to express the behavior of a system. In general, the treatment here utilizes the methods of analysis employed by texts on feedback control systems and thus is directly applicable to the synthesis problem.

THE GOVERNING LAWS OF PHYSICS

Any physical system is governed by certain laws which may or may not be formulated in mathematical expression. For the gas flow system treated herein these laws are quite simple to formulate since sufficient assumptions are made so that "classical mechanics" hold throughout. By a gas flow system is meant a connected system of ducting and restrictions through which a gas is forced to flow. General assumptions which must be made are as follows:

1. The flow is assumed to be geometrically one-dimensional and is distributed continuously and uniformly over the whole duct area, being a continuum.
2. Dissipation by radiation and thermal conductivity is neglected.
3. The fluid is a pure diatomic gas which obeys the ideal equation of state.

The pressure of a gas flowing in a duct is a function of both time and distance. If the space variable is neglected, the energy storage is said to be "lumped", and the system is said to be a "lumped parameter" system. If the space variable is not neglected, the system is a "distributed parameter" system. The equations resulting from an analysis of a "lumped parameter" system are ordinary differential equations with respect to time; those resulting from a "distributed parameter" analysis are partial differential equations with respect to both time and distance. In general, the coefficients of these differential equations are not constant but depend on the pressure, flow, and temperature in the system. The small perturbation theory is utilized in order that these coefficients may be treated as constants for a given operating point.

THE LUMPED PARAMETER EQUATIONS

Three elements are encountered in a lumped parameter system: energy storage in a volumetric capacitance, flow through a compressor, and flow through a restriction. The relations which govern flow and pressure in these elements are derived as follows:

Pressure - Weight Flow Relationship in a Volume

Consider the volume V in Fig. 1 through which a gas at pressure P and temperature Θ is flowing at a rate W.

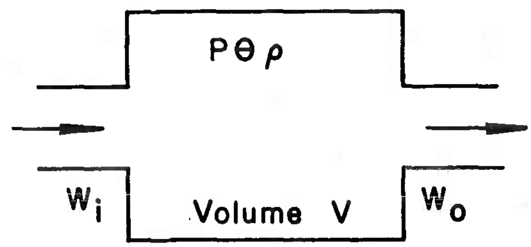

FIG. 1. SCHEMATIC OF VOLUME V

The assumption is made that temperature and pressure are instantaneously the same anywhere in the volume. Then, the rate of change of density of the gas within the volume is the difference in weight flow into and out of the volume divided by the volume. That is,

$$\frac{d\rho}{dt} = \frac{W_i - W_o}{V} . \tag{1}$$

The equation of state for a perfect gas is

$$\rho = \frac{P}{R\Theta} . \tag{2}$$

Differentiating equation (2) with respect to time and combining with equation (1) yields the general relation of pressure and flow in a volume,

$$W_i - W_o = \frac{V}{R\Theta} (\dot{P} - \frac{P}{\Theta} \dot{\Theta}) , \tag{3}$$

where the dot notation is used to denote derivative with respect to time. If the changes of state of the gas may be considered as isentropic processes, then

$$P\Theta^{-\frac{\gamma}{\gamma - 1}} = \text{const} , \tag{4}$$

and equation (3) reduces to

$$W_i - W_o = \frac{V}{\gamma R \Theta} \dot{P} . \tag{5}$$

This latter condition that the processes be isentropic can be satisfied by allowing the variables to change in small perturbations about some steady state value. That is,

$$P = P_{ss} + p$$

$$W_i = W_{ss} + w_i$$

$$W_o = W_{ss} + w_o$$

$$\Theta = \Theta_{ss} + \theta \; .$$

If these perturbations, denoted by the lower case letters, are sufficiently small the processes are essentially isentropic. Making this assumption and defining

$$C = \frac{V}{\gamma R \Theta_{ss}}$$

we obtain

$$\dot{p} = \frac{1}{C}(w_i - w_o). \tag{6}$$

On making the Laplace transformation on the variables of equation (6),

$$p(s) = \frac{1}{Cs}\left[w_i\,(s) - w_o\,(s)\right]. \tag{7}$$

The quantity C is the lumped, pneumatic capacitance of volume V. If θ is small this capacitance will be constant. It should also be noted that if the process were isothermal rather than isentropic the ratio of specific heats, γ, would be replaced by unity. Thus, C may vary by a factor from unity to 1.4 depending on the degree to which the process can be said to be isentropic.

Pressure - Weight Flow Relationship in a Compressor

In any gas flow system there must be some method of forcing the gas to flow through the volume. The device which accomplishes this is usually referred to as a compressor if it forces gas into the volume and as an exhauster if it draws gas out of the volume. The flow through this device is a function of many variables. Each compressor, or exhauster, is characterized by a unique set of performance curves which show flow as a function of the pressure ratio across

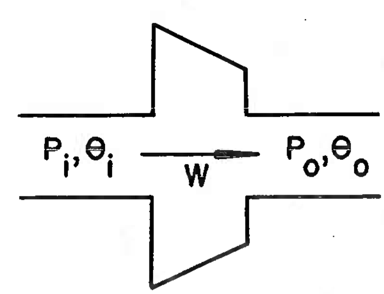

FIG. 2. SCHEMATIC OF COMPRESSOR

it for different values of inlet temperature. Thus,

$$W = f(P_i,\ P_o, \theta_i). \tag{8}$$

Taking the differential of flow yields

$$dW = \frac{\partial W}{\partial P_i} dP_i + \frac{\partial W}{\partial P_o} dP_o + \frac{\partial W}{\partial \Theta_i} d\Theta_i . \tag{9}$$

Making use of the theory of small perturbations so that linearity is assured

$$w = k_e p_1 + k_c p_2 + k_t\ \theta , \tag{10}$$

where the following definitions have been made:

$$w = dW \qquad k_e = \frac{\partial W}{\partial P_i}$$

$$p_i = dP_i \qquad k_c = \frac{\partial W}{\partial P_o}$$

$$p_o = dP_o$$

$$\theta = d\Theta_i \qquad k_t = \frac{\partial W}{\partial \Theta_i} .$$

The constants denoted by k must then be evaluated at each operating condition from performance curves.

Pressure - Weight Flow Relationship in a Restriction

Control of pressure in a gas flow system is accomplished by control of flow at some point; this in turn is accomplished by controlling the area of some restriction, usually a valve of some configuration, at that point. In general, the flow through a restriction is dependent on upstream pressure, downstream pressure, temperature, and the open area of the restriction. That is,

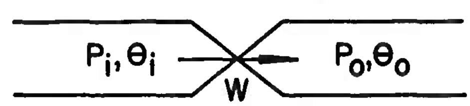

FIG. 3. SCHEMATIC OF RESTRICTION

$$W = f(P_i , P_o , \Theta_i , A) . \tag{11}$$

Taking the differential and using the theory of small perturbations as before,

$$w = k_i\, p_i + k_o\, p_o + k_\theta\ \theta + k_a\, a, \tag{12}$$

where the following definitions have been made:

$$w = dW \qquad k_i = \frac{\partial W}{\partial P_i}$$

$$p_i = dP_i \qquad k_o = \frac{\partial W}{\partial P_o}$$

$$p_o = dP_o \qquad k_\theta = \frac{\partial W}{\partial \Theta_i}$$

$$\theta = d\Theta_i \qquad k_a = \frac{\partial W}{\partial A}$$

$$a = dA \, .$$

These constants depend upon knowledge of the valve's pressure-flow-temperature relationship, which may be expressed either by families of curves or by approximate mathematical expressions. A method is given in Appendix 1 whereby these constants may be determined for some types of valves. This method, with sufficient experimental data, can be extended to any valve configuration.

THE DISTRIBUTED PARAMETER EQUATIONS

Consider an element dx of a straight duct of constant cross-sectional area A_o in which a compressible fluid of density ρ and at pressure P is flowing at a rate W and velocity U as shown in Fig. 4. The quantities are shown at any instant of time t. The partial differential equations which describe the pressure-flow relationship are derivable from an expression of the continuity equation and a net force equation.

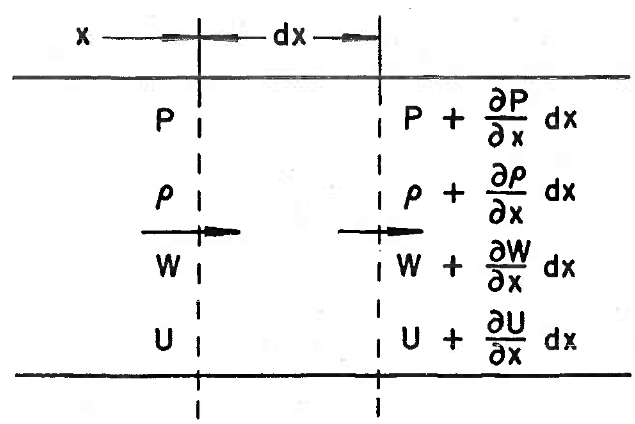

FIG. 4. SCHEMATIC OF INCREMENT OF FLOW IN A DUCT

The continuity equation states that the rate of decrease of weight within the element of volume $A_o dx$ is equal to the net rate of weight flow in the element,

$$-\frac{\partial W^*}{\partial t} = (W + \frac{\partial W}{\partial x} dx) - W \quad . \tag{13}$$

The weight of gas in the volume is the density times the volume and hence,

$$W^* = (\rho + \frac{\partial \rho}{\partial x} dx) A_o dx .$$

Neglecting the higher order terms yields

$$\frac{\partial W^*}{\partial t} = \frac{\partial \rho}{\partial t} A_o dx . \tag{14}$$

Equating (14) and (13) yields

$$-\frac{\partial W}{\partial x} = A_o \frac{\partial \rho}{\partial t} . \tag{15}$$

The process is assumed isentropic as in the case of lumped parameter equations. For isentropic processes all fluid states have the same entropy. This fact can be stated as follows (see Ref. 3):

$$\frac{\partial P}{\partial \rho \begin{pmatrix}\text{constant}\\ \text{entropy}\end{pmatrix}} = \frac{dP}{d\rho} = \frac{u^2}{g} , \tag{16}$$

where u is the velocity of propagation of a pressure disturbance. The equation of continuity can then be stated as

$$-\frac{\partial W}{\partial x} = \frac{A_o g}{u^2} \frac{\partial P}{\partial t} = c \frac{\partial P}{\partial t} . \tag{17}$$

The net force acting on the fluid within the elemental volume is equal to the mass of the fluid times acceleration. The mass of fluid in the volume is

$$\frac{W^*}{g} = \frac{1}{g}(\rho + \frac{\partial \rho}{\partial x} dx) A_o dx .$$

The velocity of the incremental volume is

$$U = \frac{W}{A_o \rho} + \frac{\partial}{\partial x} (\frac{W}{A_o \rho}) dx$$

and the corresponding acceleration is

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial t} (\frac{W}{A_o \rho}) + \frac{\partial}{\partial t} \left[\frac{\partial}{\partial x} (\frac{W}{A_o \rho}) dx \right] .$$

The net force is then

$$PA_o - \left[P + \frac{\partial P}{\partial x} dx \right] A_o = \left[\frac{1}{g} (\rho + \frac{\partial \rho}{\partial x} dx) A_o dx \right] \left\{ \frac{\partial}{\partial t} (\frac{W}{A_o \rho}) + \frac{\partial}{\partial t} \left[\frac{\partial}{\partial x} (\frac{W}{A_o \rho}) dx \right] \right\} . \tag{18}$$

Upon simplifying equation (18) and neglecting the higher order terms in dx, equation (18) reduces to

$$-\frac{\partial P}{\partial x} = \frac{\rho}{A_o g} \frac{\partial\left(\frac{W}{\rho}\right)}{\partial t} = \frac{1}{gA_o} \frac{\partial W}{\partial t} - \frac{W}{A_o \rho g} \frac{\partial \rho}{\partial t} . \tag{19}$$

In order to handle equation (19) advantageously, it is necessary to assume that total velocity of the gas stream is small. Then, the last term of equation (19) can be neglected and

$$-\frac{\partial P}{\partial x} = \frac{1}{A_o g} \frac{\partial W}{\partial t} . \tag{20}$$

If the quantities are treated as small perturbations about a steady state value as in the lumped parameter treatment, then equations (17) and (20) become

$$-\frac{\partial w}{\partial x} = c \frac{\partial p}{\partial t} \tag{21}$$

$$-\frac{\partial p}{\partial x} = h \frac{\partial w}{\partial t} . \tag{22}$$

There also occurs a loss in pressure with distance due to various resistances. To include this term apparently contradicts the assumption of isentropic processes made previously. However, change in enthropy with distance does not preclude an isenthropic compression in an incremental element. This pressure drop along the duct can be taken as being a function of flow. Hence, over a small incremental change (again using the small perturbations):

$$-\frac{dP}{dx} = \frac{dP}{dW} \frac{dW}{dx} ,$$

or

$$-\frac{dp}{dx} = r\, w ; \; r = \frac{\Delta P}{\Delta W \Delta x} . \tag{23}$$

Combining this additional pressure drop with that of equation (22) yields

$$-\frac{\partial p}{\partial x} = h \frac{\partial w}{\partial t} + r\, w . \tag{24}$$

The quantities c, h and r are usually referred to as the distributed pneumatic capacitance, inertance and resistance, respectively.

A SIMPLE SYSTEM

The use of the foregoing physical laws can best be illustrated by means of an example. It is in the response of a system that the correlation between the lumped and distributed parameter descriptions of system dynamics can best be seen. The treatment will also be useful since the simple system to be used is a sufficient description of many processes.

The system to be studied is shown in Fig. 5. In Fig. 5(a) the system is represented as a lumped parameter system. A gas flows into volume V through some restriction or device at a rate W_i and out through another device at a rate W_o, with a pressure P and a temperature Θ. In Fig. 5(b) the system is represented as a distributed parameter system. The gas flows into a straight duct, of Length L and of cross-sectional area A_o (such that $V = A_o L$), from the same source as in 5(a) and out through the same device as in 5(a), maintaining the same steady state pressure and temperature.

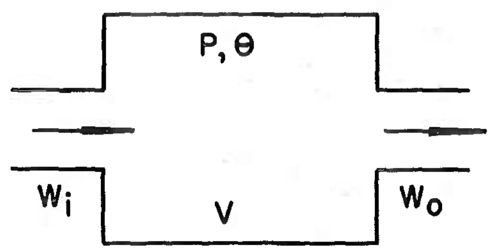

FIG. 5 (a). LUMPED PARAMETER SYSTEM

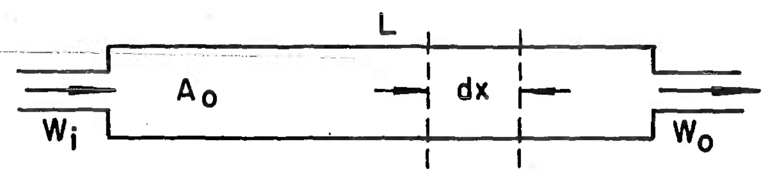

FIG. 5 (b). DISTRIBUTED PARAMETER SYSTEM

The flow into the system is an independent variable. The flow out of the system is considered to be proportional only to the pressure P. Temperature is considered constant in the evaluation of the capacitance term.

PRESSURE RESPONSE TO AN INFLOW DISTURBANCE AS DETERMINED BY THE LUMPED PARAMETER EQUATIONS

Two equations describe this system according to the linearized, small perturbation theory presented previously:

$$p(s) = \frac{1}{Cs}\left[w_i(s) - w_o(s)\right] \text{ and} \tag{25}$$

$$w_o(s) = k_o\, p(s), \tag{26}$$

where $C = \dfrac{V}{\gamma R\Theta}$ and $k_o = \dfrac{\partial W_o}{\partial P} = \dfrac{W_{ss}}{P_{ss}}$. Combining these equations yields the system transfer function,

$$\frac{p(s)}{w_i(s)} = \frac{1/k_o}{\dfrac{C}{k_o}s + 1} = \frac{K}{\tau s + 1}. \tag{27}$$

The gain of the system is $K = \dfrac{P_{ss}}{W_{ss}}$ and the time constant is $\tau = \dfrac{C}{k_o} = \dfrac{V P_{ss}}{\gamma R\Theta W_{ss}}$.

Frequency Response

Of interest in the synthesis of control systems is the frequency response of the system to be controlled. Replacing s by $j\omega$ the frequency response of transfer function (27) can be determined. The straight line approximation of this frequency response is shown in non-dimensionalized form in Fig. 6.

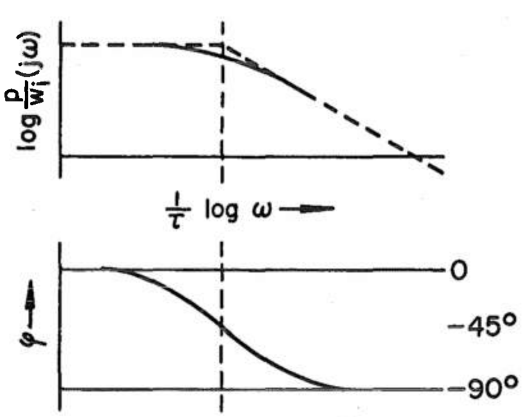

FIG. 6. LUMPED PARAMETER FREQUENCY RESPONSE

Indicial Response

The usual manner of showing the time response of a system is to express the time solution to a step input, sometimes referred to as the indicial response. In mathematical form this is

$$k_o \left| \frac{p}{w} \right| = 1 - e^{\left(\frac{-t}{\tau} \right)} . \tag{28}$$

The graph is shown in Fig. 7.

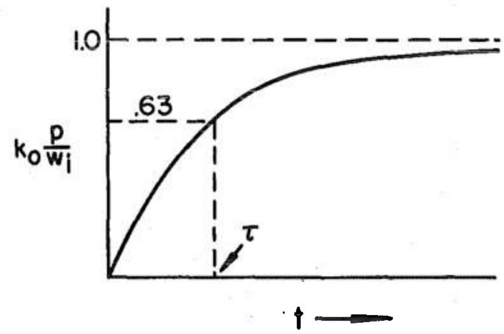

FIG. 7. LUMPED PARAMETER INDICIAL RESPONSE

PRESSURE RESPONSE TO AN INFLOW DISTURBANCE AS DETERMINED BY THE DISTRIBUTED PARAMETER EQUATIONS

The solution of equations (21) and (24) to sinusoidal and indicial forcings yields the distributed parameter response of the example system. These two solutions are sufficiently dissimilar to require separate treatments. The first uses the standard $j\omega$ operator, whereas the latter must employ the Laplace operator.

Frequency Response

Suppose the quantities w and p of equations (21) and (24) to be small sinusoidal perturbations. The impedance notation commonly used in the study of steady state electrical alternating current can then be used and the equations become

$$\frac{\partial w}{\partial x} = c \frac{\partial p}{\partial t} = j\omega c p = yp \tag{29}$$

$$\frac{\partial p}{\partial x} = rw + h \frac{\partial w}{\partial t} = (r + j\omega h) w = zw , \tag{30}$$

where y and z are the pneumatic admittance and impedance, respectively, while

$j\omega$ is the complex frequency operator. It should be noted also that equations (29) and (30) differ from equations (21) and (24) by a minus sign since here it is convenient to measure distance from the outlet end of the system.

The analogy between the distributed parameter gas flow system and an electrical transmission line can be observed through examination of equations (29) and (30).

Taking the derivative of equation (30) with respect to distance yields

$$\frac{d^2 p}{dx^2} = z \frac{dw}{dx} = zyp$$

or

$$\frac{d^2 p}{dx^2} - zyp = 0 \tag{31}$$

A solution to this equation is

$$p_x = A^* e^{x\sqrt{zy}} + B^* e^{-x\sqrt{zy}}, \tag{32}$$

and from equation (29)

$$w_x = A^* \sqrt{\frac{y}{z}}\, e^{x\sqrt{zy}} - B^* \sqrt{\frac{y}{z}}\, e^{-x\sqrt{zy}}. \tag{33}$$

where A^* and B^* are to be determined from boundary conditions. For the system under consideration, at $x = 0$, $p_x = p_o$ and $w_x = w_o$. After evaluating A^* and B^* and simplifying, the solution can be written as

$$p_x(j\omega) = p_o \cosh x\sqrt{zy} + w_o\sqrt{\frac{z}{y}} \sinh x\sqrt{zy} \tag{34}$$

$$w_x(j\omega) = w_o \cosh x\sqrt{zy} + p_o\sqrt{\frac{z}{y}} \sinh x\sqrt{zy}. \tag{35}$$

As mentioned previously the disturbance to the system is a flow oscillation, w_i, at $x = L$. Hence, we desire the pressure response to this disturbance at any position in the ducting. Making the definitions

$$Z = \frac{dP_o}{dW_o} = \frac{p_o}{w_o} = \frac{P_o}{W_o},$$

$$\sqrt{zy} = a + j\beta,$$

$$\sqrt{\frac{z}{y}} = \frac{1}{\omega c}(\beta - ja);$$

where

$$\beta = \sqrt{\tfrac{1}{2}\,\omega c\,[\omega h + \sqrt{\omega^2 h^2 + r^2}]}\ ,$$

$$a = \frac{\omega r c}{2\beta};$$

and using certain trigonometric and hyperbolic identities, equations (34) and (35) may be manipulated to give

$$\frac{p_x(j\omega)}{w_i(j\omega)} = \frac{f_1 + j f_2}{f_3 + j f_4} = G_x \underline{/\phi_x}\ , \qquad (36)$$

where

$$G_x = \sqrt{\frac{f_1^2 + f_2^2}{f_3^2 + f_4^2}}\ , \qquad (37)$$

$$\phi_x = \mathrm{Tan}^{-1}\frac{f_2}{f_1} - \mathrm{Tan}^{-1}\frac{f_4}{f_3}\ , \qquad (38)$$

and the functions of $j\omega$ are given by

$$f_1 = Z\cos(\beta x)\cosh(ax) + \frac{\beta}{\omega c}\cos(\beta x)\sinh(ax) + \frac{a}{\omega c}\sin(\beta x)\cosh(ax)$$

$$f_2 = Z\sin(\beta x)\sinh(ax) + \frac{\beta}{\omega c}\sin(\beta x)\cosh(ax) - \frac{a}{\omega c}\cos(\beta x)\sinh(ax)$$

$$f_3 = \cos(\beta L)\cosh(aL) + \frac{Z\omega c\beta}{a^2+\beta^2}\cos(\beta L)\sinh(aL) - \frac{Z\omega c a}{a^2+\beta^2}\sin(\beta L)\cosh(aL)$$

$$f_4 = \sin(\beta L)\sinh(aL) + \frac{Z\omega c\beta}{a^2+\beta^2}\sin(\beta L)\cosh(aL) + \frac{Z\omega c a}{a^2+\beta^2}\cos(\beta L)\sinh(aL)\ .$$

Thus, equations (36), (37) and (38) give the frequency response of the system of Fig. 5 (b) at any position x, where x is measured from the outlet.

Considerable simplification of this complicated frequency response function results for the special case of zero resistance ($r = 0$). Then $\beta = \omega/u$, $a = 0$ and the functions of $j\omega$ become

$$f_1 = Z\cos\beta x = Z\cos\frac{\omega}{u}x \qquad f_3 = \cos\beta L = \cos\frac{\omega}{u}L$$

$$f_2 = \frac{\beta}{\omega c}\sin\beta x = \frac{1}{uc}\sin\frac{\omega}{u}x \qquad f_4 = Z\frac{\omega c}{\beta}\sin\beta L = Z\,cu\sin\frac{\omega}{u}L$$

At $x = 0$, $f_1 = Z$ and $f_2 = 0$. The functions f_3 and f_4 range from maxima to minima at $\beta L = n\pi/2$, where n is any integer and for $r = 0$, $\beta = \frac{\omega}{u}$, owing to the nature of the trigonometric functions. This occurs at such values of frequency, ω, such that L is equal to some multiple of a quarter wave length. That is, at

$L = \frac{n}{4}\lambda = \frac{n\pi u}{2\omega_c}$, and $\omega_c = \frac{n\pi u}{2L}$ where ω_c is denoted as a "critical" frequency. From this alone, the maxima and minima of the frequency response can be determined. This treatment yields the same information when applied to the position $x = L$. Thus, the frequency response plots of the distributed parameter system can be approximately determined as shown in Fig. 8.

Indicial Response

The methods of the transformation calculus can be used to obtain the pressure response to a step disturbance in flow. By obtaining the Laplacian transform of response, finding the singular points and residues of the transformed equation and applying the "Inversion Theorem" (see Ref. 4), the time solution can be evaluated. This process will be applied in this section to equations (21) and (24), since here it is convenient to measure x from the inlet.

Differentiating equations (21) and (24) with respect to distance and time respectively yields

$$-\frac{\partial^2 w}{\partial x^2} = c\,\frac{\partial^2 p}{\partial t\,\partial x}, \tag{39}$$

$$-\frac{\partial^2 p}{\partial x \partial t} = h\,\frac{\partial^2 w}{\partial t^2} + r\,\frac{\partial w}{\partial t}; \tag{40}$$

or, upon combining

$$\frac{\partial^2 w}{\partial x^2} = hc\,\frac{\partial^2 w}{\partial t^2} + rc\,\frac{\partial w}{\partial t}. \tag{41}$$

In a similar manner, differentiating equations (21) and (24) with respect to time and distance, respectively, and combining, yields

$$\frac{\partial^2 p}{\partial x^2} = hc\,\frac{\partial^2 p}{\partial t^2} + rc\,\frac{\partial p}{\partial t}. \tag{42}$$

If, initially,

$$w(o,x) = p(o,x) = \left(\frac{\partial w}{\partial t}\right)_{t=o} = \left[\frac{\partial p}{\partial t}\right]_{t=o} = 0,$$

then the Laplace transformation of equations (41) and (42) yields

$$\frac{d^2 w(s,x)}{dx^2} = (hc\,s + rc)\,s\,w(s,x), \tag{43}$$

$$\frac{d^2 p(s,x)}{dx^2} = (hc\,s + rc)\,s\,p(s,x). \tag{44}$$

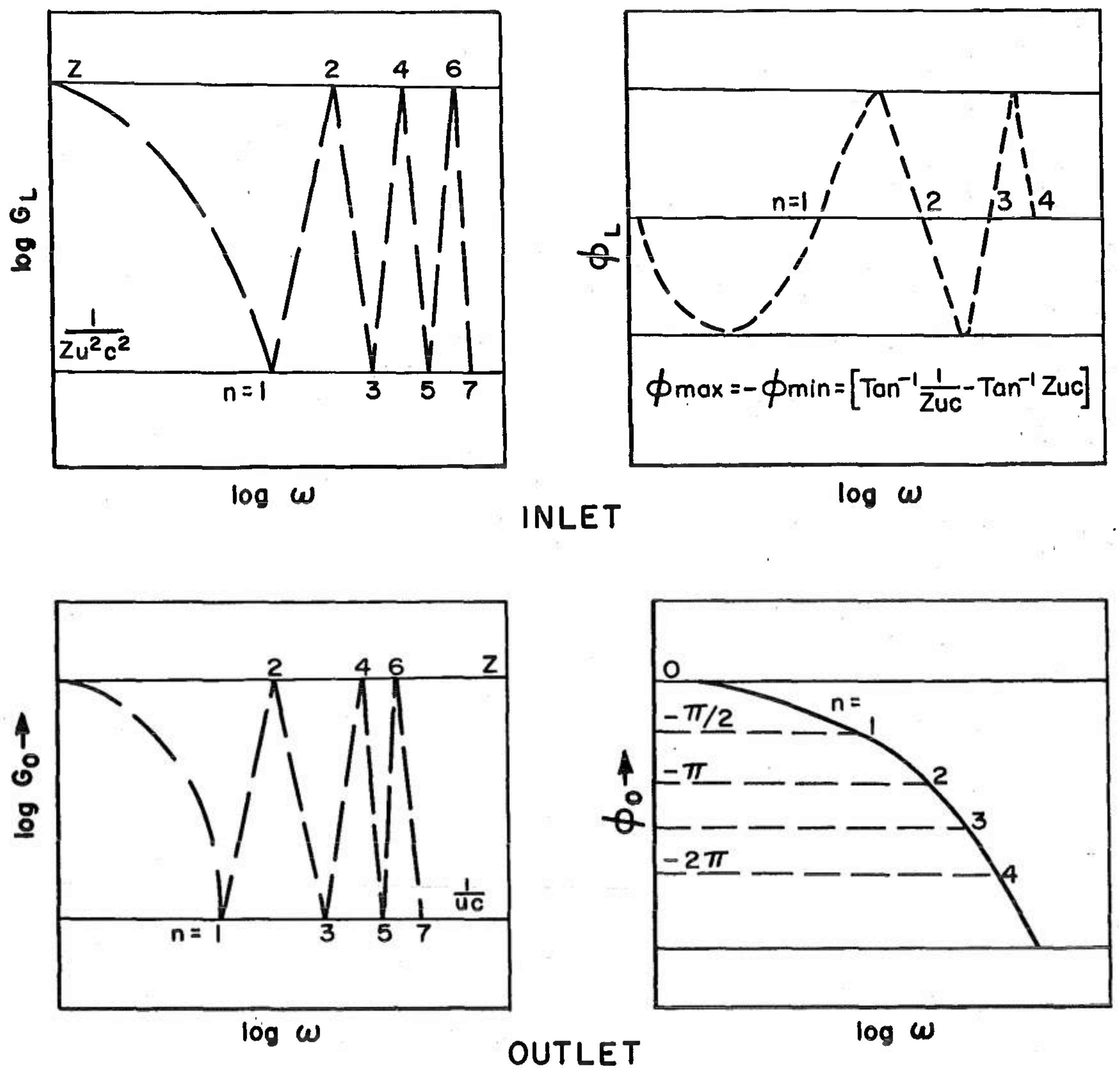

FIG. 8. DISTRIBUTED PARAMETER FREQUENCY RESPONSE

These equations have the solutions

$$w(s,x) = C^* e^{(hcs^2 + rcs)^{1/2} x} + D^* e^{-(hcs^2 + rcs)^{1/2} x} \tag{45}$$

$$p(s,x) = E^* e^{(hcs^2 + rcs)^{1/2} x} + F^* e^{-(hcs^2 + rcs)^{1/2} x} . \tag{46}$$

The total length of the system is taken as L. At $x = L$, $w(s) = w_o(s)$, and $p(s) = p_o(s)$. In addition, the "terminal impedance" is defined by $Z = \frac{p_o(s)}{w_o(s)}$, and it is also expedient to make the definition

$$Q = \left(\frac{r + hs}{hs}\right)^{1/2} .$$

Upon evaluating C^*, D^*, E^* and F^* as functions of s, equations (45) and (46) become

$$w(s,x) = \frac{w_o(s)}{2Q\sqrt{\frac{h}{c}}}\left[\left(Z + Q\sqrt{\frac{h}{c}}\right) e^{Q\sqrt{hc}\, s(L-x)} - \left(Z - Q\sqrt{\frac{h}{c}}\right) e^{-Q\sqrt{hc}\, s(L-x)}\right] \tag{47}$$

$$p(s,x) = \frac{w_o(s)}{2}\left[\left(Z + Q\sqrt{\frac{h}{c}}\right) e^{Q\sqrt{hc}\, s(L-x)} + \left(Z - Q\sqrt{\frac{h}{c}}\right) e^{-Q\sqrt{hc}\, s(L-x)}\right] . \tag{48}$$

The transfer function of pressure to flow disturbance can then be written as

$$\frac{p(s,x)}{w(s,x)} = \sqrt{\frac{h}{c}}\, Q \, \frac{\left(Z + Q\sqrt{\frac{h}{c}}\right) e^{Q\sqrt{hc}\, s(L-x)} + \left(Z - Q\sqrt{\frac{h}{c}}\right) e^{-Q\sqrt{hc}\, s(L-x)}}{\left(Z + Q\sqrt{\frac{h}{c}}\right) e^{Q\sqrt{hc}\, s(L-x)} - \left(Z - Q\sqrt{\frac{h}{c}}\right) e^{-Q\sqrt{hc}\, s(L-x)}} . \tag{49}$$

If the disturbance occurs at $x = o$, the transfer function can be written as

$$\frac{p(s,x)}{w(s,o)} = \sqrt{\frac{h}{c}}\, Q \, \frac{\left(\dfrac{Z + Q\sqrt{\frac{h}{c}}}{Z - Q\sqrt{\frac{h}{c}}}\right) e^{Q\sqrt{hc}\, s(2L-x)} + e^{Q\sqrt{hc}\, s x}}{\left(\dfrac{Z + Q\sqrt{\frac{h}{c}}}{Z - Q\sqrt{\frac{h}{c}}}\right) e^{2Q\sqrt{hc}\, s L} - 1} . \tag{50}$$

It is convenient, here, to introduce the relation

$$e^{2\sigma\pi} = \frac{Z - \sqrt{\frac{h}{c}}\, Q}{Z + \sqrt{\frac{h}{c}}\, Q} .$$

Then

$$\frac{p(s,x)}{w(s,o)} = \sqrt{\frac{h}{c}}\, Q \; \frac{e^{Q\sqrt{hc}\; s(2L-x) \;-\; 2\pi\sigma} + e^{Q\sqrt{hc}\; s x}}{e^{2Q\sqrt{hc}\, sL \;-\; 2\sigma\pi} - 1}. \tag{51}$$

It is now expedient, if the mathematics is to be kept simple, to examine the case $r=0$, therein making Q equal to unity. The solution is desired to a step in flow; that is, $w_i\,(s) = w_i\,\frac{1}{s}$. The poles of equation (51) are (1) $s = 0$ and (2) the roots of $e^{2s\sqrt{hc}\;L \,-\, 2\sigma\pi} = e^{2\pi jk} = 1$, which are $s = \frac{\pi}{\sqrt{hc}\;L}\;(\sigma + jk)$, where $k = 0, \pm 1, \pm 2,$ ---

The residues at these singularities are then

1. $R_1^* = Z$

2. $$R_2^* = \frac{1}{2\pi}\sqrt{\frac{h}{c}}\; \frac{e^{\frac{\pi x}{L}\,(\sigma + jk)} + e^{-\pi\frac{X}{L}\,(\sigma + jk)}}{\sigma + jk}$$

$$= \frac{1}{\pi}\sqrt{\frac{h}{c}}\; \frac{\sigma - jk}{\sigma^2 + k^2}\; e^{\frac{jk\pi x}{L}} \cosh \frac{\pi\sigma x}{L},$$

where the simplification of R_2^* is obtained by using $^{2\pi jkx/L} = (e^{2\pi jk})^{\frac{x}{L}} = 1.$

The inverse of the transform is obtained by summing the product $R^* e^{st}$ at each singularity. Then

$$\frac{p(x,t)}{w_i} = \sum R^* e^{st} = Z + \sum_{k=-\infty}^{k=+\infty} \frac{1}{\pi}\sqrt{\frac{h}{c}}\; \frac{\sigma - jk}{\sigma^2 + k^2}\; e^{j\pi kx/L} \cosh \frac{\pi\sigma x}{L}\; e^{\frac{\pi}{\sqrt{hc}\,L}(\sigma x jk)\, t}. \tag{52}$$

By introducing the identify

$$e^{j\pi kx/L} = (-1)^{\eta k}\, e^{-j\eta\pi k}\, e^{j\pi kx/L} = (-1)^k\, e^{\frac{j\pi k}{L}(x - \eta L)} \qquad \eta = 1, 3, 5 - - - .$$

equation (52) becomes

$$\frac{p(x,t)}{|w_i|} = Z + \frac{1}{\pi}\sqrt{\frac{h}{c}}\, \cosh\frac{\pi\sigma x}{L}\; e^{\frac{\pi\sigma t}{\sqrt{hc}\,L}} \sum_{k=-\infty}^{k=+\infty} \frac{\sigma - jk}{\sigma^2 + k^2}\, (-1)^k\; e^{\frac{jk\pi}{\sqrt{hc}\,L}\,[t + \sqrt{hc}\,(x - \eta L)]} \tag{53}$$

Expanding equation (53) yields

$$\frac{p(x,t)}{|w_i|} = Z + \frac{1}{\pi}\sqrt{\frac{h}{c}}\cosh\frac{\pi\sigma x}{L}\, e^{\frac{\pi\sigma t}{\sqrt{hc}\,L}} \left\{ 2\sigma\left[\frac{1}{2\sigma^2} + \sum_{1}^{\infty}(-1)^k \frac{\cos k\theta^*}{\sigma^2+k^2}\right] + 2\left[\sum_{1}^{\infty}(-1)^k \frac{k\sin k\theta^*}{\sigma^2+k^2}\right]\right\}, \tag{54}$$

where $\theta^* = \frac{\pi}{\sqrt{hc}\,L}[\,t + \sqrt{hc}\,(x-\eta L)\,]$. This form brings to attention two Fourier series identities which can be written as

$$\cosh\sigma\theta^* = \frac{2\sigma}{\pi}\sinh\sigma\pi\left[\frac{1}{2\sigma^2} + \sum_{1}^{\infty}(-1)^k\frac{\cos k\theta^*}{\sigma^2+k^2}\right],$$

$$\sinh\sigma\theta^* = -\frac{2}{\pi}\sinh\sigma\pi\left[\sum_{1}^{\infty}(-1)^k\frac{k\sin k\theta^*}{\sigma^2+k^2}\right],$$

where $-\pi < \theta^* < \pi$. Substitution of these identities (Ref. 5) into equation (54) yields, after simplification,

$$\frac{p(x,t)}{|w_i|} = Z + \sqrt{\frac{h}{c}}\, e^{\frac{\pi\sigma t}{\sqrt{hc}\,L}}\, \frac{\cosh\frac{\pi\sigma x}{L}}{\sinh\sigma\pi}\, e^{-\sigma\theta^*}. \tag{55}$$

The boundary placed on θ^* requires that for each η, t be confined to the region

$$\sqrt{hc}\left[L(\eta-1) - x\right] < t < \sqrt{hc}\left[L(\eta+1) - x\right]$$

Allowing $\eta = 2n+1$ where $n = 0, 1, 2, ---$ and remembering the definition of $2\sigma\pi$ for $Q = 1$, equation (55) further reduces to

$$\frac{p(x,t)}{|w_i|} = Z - \frac{1}{2}\left[Z - \sqrt{\frac{h}{c}}\right]\left[1 + \left(\frac{Z+\sqrt{\frac{h}{c}}}{Z-\sqrt{\frac{h}{c}}}\right)^{\frac{x}{L}}\right]\left[\frac{Z-\sqrt{\frac{h}{c}}}{Z+\sqrt{\frac{h}{c}}}\right]^n. \tag{56}$$

A further reduction results from letting $\delta = \frac{\sqrt{\frac{h}{c}}}{Z}$,

$$\frac{p(x,t)}{Z\,|w_i|} = 1 - \frac{1}{2}(1-\delta)\left[1 + \left(\frac{1+\delta}{1-\delta}\right)^{\frac{x}{L}}\right]\left[\frac{1-\delta}{1+\delta}\right]^n \tag{57}$$

where $n = 0, 1, 2, ----,$ and

$$\sqrt{hc}\ (2\,nL - x) < t < \sqrt{hc}\left[2\,(n+1)\,L - x\right].$$

For the two positions of interest, $x = o$, and $x = L$, the solutions are as follows:

for $x = o$

$$\frac{p(o,t)}{Z\,|w_i|} = 1 - \frac{(1-\delta)^{n+1}}{(1+\delta)^n}, \tag{58}$$

$$2n < \frac{t}{\sqrt{hc}\,L} < 2(n+1)\ .$$

for $x = L$

$$\frac{p(L,t)}{Z\,|w_i|} = 1 - \left(\frac{1-\delta}{1+\delta}\right)^n$$

$$(2\,n-1) < \frac{t}{\sqrt{hc}\,L} < (2\,n+1)\ . \tag{59}$$

CORRELATION OF LUMPED AND DISTRIBUTED PARAMETER ANALYSES

An examination of the distributed parameter frequency response discloses that for small values of the frequency variable ω the lumped and distributed parameter responses are equivalent. In order to show this, equation (27) can be written as

$$\frac{p\,(j\omega)}{w_i\,(j\omega)} = \frac{\dfrac{P}{\overline{W}}}{\dfrac{VP}{\gamma R\Theta\overline{W}}\,(j\omega) + 1}\ . \tag{60}$$

For the special case $r = o$ equation (36) can be written as

$$\frac{p(j\omega)}{w_1(j\omega)} = \frac{f_1 + jf_2}{f_3 + jf_4} = \frac{Z\cos\beta x + j\dfrac{\beta}{\omega c}\sin\beta x}{\cos\beta L + j\,Z\dfrac{\omega c}{\beta}\sin\beta L}$$

For values of $\omega << \frac{u}{L}$, $\cos\beta_x = 1$, and $\sin\beta_x = \beta_x$ because $\beta x \leq \beta L = \frac{\omega L}{u} << 1.$

Thus,

$$\frac{p\,(j\omega)}{w_i(j\omega)} = \frac{Z + j\dfrac{\beta}{\omega c}\beta x}{1 + j\dfrac{Z\,\omega c}{\beta}\beta L} = \frac{Z + j\,\dfrac{\beta^2 x}{\omega c}}{1 + j\,Z\,\omega cL}\ . \tag{61}$$

Now, remembering that $\beta = \frac{\omega}{u}$ for $r = o$, $Z = \frac{P}{W}$, $c = \frac{A_o g}{u^2}$, $u^2 = \gamma g R \Theta$ and $LA_o = V$, equation (61) can be reduced to

$$\frac{p\ (j\omega)}{w_i\ (j\omega)} = \frac{\frac{P}{W} + j\ \frac{\beta x}{uC}}{\frac{PV}{\gamma R \Theta W}\ (j\omega) + 1}\ . \tag{62}$$

Since $\beta x \ll 1$ and $\frac{1}{uc} < \frac{P}{W}$ (Fig. 8), equation (62) reduces to equation (27).

The deviation of the distributed parameter frequency response from the lumped parameter begins to become excessive for frequencies just below the first critical frequency ($\omega_c = \frac{\pi u}{2L}$). As previously noted the break in the lumped constant frequency response occurs at $\omega_b = \frac{1}{\tau} = \frac{\gamma R \Theta W}{PV}$. The ratio of these is

$$\frac{\omega_b}{\omega_c} = \frac{\frac{\gamma R \Theta W}{PV}}{\frac{\pi u}{2L}}\ . \tag{63}$$

This expression reduces to

$$\frac{\omega_b}{\omega_c} = \frac{2\gamma}{\pi} M \tag{64}$$

where M is the Mach number of the gas flowing in the system. The constant $2\gamma/\pi$ is about unity so that the ratio is about equal to the Mach number. If the ratio is small the deviation occurs far down on the frequency response gain curve; if the ratio is large the deviation occurs early on the frequency response plot.

Examination of the indicial response indicates that the solution (56) can be reduced to the lumped parameter solution (28) by the following treatment. Suppose the ducting length L tends to zero while the volume $L\ A_o$ remains fixed. Then $\frac{x}{L}$ tends to unity, A_o tends to infinity, $\sqrt{\frac{h}{c}} = \frac{u}{A_o g}$ tends to zero, and thus

δ approaches zero. The limit on time may then be written as

$$(2n - \frac{x}{L}) < \frac{t}{\sqrt{hc}\,L} < \left[2(n+1) - \frac{x}{L}\right] . \tag{65}$$

For any fixed time t as L approaches zero, n tends to $\frac{t}{2\,hc\,L}$. Then

$$\frac{p(x,t)}{Z\,|w_i|} = 1 - \left(\frac{1-\delta}{1+\delta}\right)^{\frac{t}{2\sqrt{hc}\,L}} . \tag{66}$$

Now, remembering that $\lim\limits_{x \to 0} (1 + bx)^{\frac{1}{x}} = e^b$

it is convenient to write

$$\frac{p(x,t)}{Z\,|w_i|} = 1 - \left[\left(\frac{1-\delta}{1+\delta}\right)^{\frac{1}{\delta}}\right]^{\frac{\delta t}{2\sqrt{hc}\,L}} ;$$

then

$$\left(\frac{1-\delta}{1+\delta}\right)^{\frac{1}{\delta}} = (1+\delta)^{-\frac{1}{\delta}} (1-\delta)^{\frac{1}{\delta}} .$$

Letting $b = 1$ and $b = -1$, respectively.

$$\lim_{\delta \to 0} \left(\frac{1-\delta}{1+\delta}\right)^{\frac{1}{\delta}} = e^{-2} ,$$

$$\frac{p(x,t)}{Z\,|W_i|} = 1 - e^{-\frac{\frac{t}{\sqrt{hc}\,L}}{\delta}} = 1 - e^{-\frac{t}{\frac{\gamma R \Theta W}{PV}}} , \tag{67}$$

since the quantity $\frac{\sqrt{hc}\,L}{\delta}$ can be reduced to $\frac{PV}{\gamma R \Theta W}$, which is the time constant τ , equation (28). Thus, the distributed parameter solution can be reduced to the lumped parameter solution if the length of the ducting is made small and the area large.

OBSERVATIONS ON THE RESPONSE EQUATIONS

There are a number of interesting observations that can be made on the analysis so far. The treatment has been chiefly analytical without regard for physical concepts. This section is devoted to observations on the nature of the different properties of gas flow systems as revealed in this analysis.

NATURE OF THE FORCING FUNCTION

The forcing function, or disturbance, considered in the illustrative example was a change in flow into the system at the inlet. This disturbance is the major forcing for many systems, including the exhaust system of an engine test plant. One might as easily be concerned with disturbances in inlet or outlet pressure, in valve area, or in parameters such as the volume of a bellows. In any case the method of treatment would be similar so that the analysis might be used on such varied problems as pressure response of instrument lines and transducers, acoustical design of mufflers, analysis of pneumatic control elements, and the dynamic analysis of wind tunnel performance. The specific discussion is limited to changes in flow or valve area.

NATURE OF THE RESPONSE

The response desired of gas flow systems is nearly always the pressure change due to some forcing. In system synthesis for control purposes the steady-state response to sinusoidal forcing and the time response to step function forcing are usually sufficient to adequately describe control characteristics. Depending on the nature of the problem, varying degrees of completeness of the response may be needed. For some problems only the lumped parameter transfer function may be required. For others the complete distributed response to a step or other function may be required.

The Lumped Parameter Transfer Function

In the description of systems, perhaps the most informative knowledge is an expression of the linear, non-dimensionalized transfer function. One sees in this expression the sensitivity of the system (gain), the degree of complexity, or order, of the system, the magnitude of the time constants, and the degree of stability to be expected. Fairly accurate knowledge of these is in many instances sufficient for the design of control systems.

Formulation of the lumped parameter transfer function is illustrated by the system shown in Appendix 2. The equations governing lumped parameter are shown at their respective locations, and the transfer function resulting from the combination of these equations is also shown. In the system of Appendix 2, a flow of gas W_i enters the volume V_1, passes through a control valve system into volume V_2, and is pumped out by an exhauster which has a recirculation line around it. The control valve system consists of a throttle valve and a bleed valve. The inlet flow is an independent quantity which deviates from a steady state. The pressure response to this deviation is desired. The simultaneous solution of all the transformed equations which describe each system element yields the transfer function.

There are several interesting properties of this transfer function. The system contains two energy storage elements, the two volumes; and hence a second order denominator in the transfer function is assured. The system constants are all real, positive numbers; and this assures, for the second order system, that the system will be stable. Furthermore, when the transfer function is written in the form shown, where ω_n is the natural frequency and ζ is the damping ratio, it can be shown that ζ is always greater than unity, thus assuring that the system is over-damped. This also assures that the denominator has two real roots and can be expressed as the product of two first order lags, $(\tau_3\ s+1)\ (\tau_4\ s+1)$. Now it can also be shown that $\tau_3 \leq \tau_1 \leq \tau_4$ and that $\tau_3 \leq \tau_2 \leq \tau_4$. This guarantees that the log magnitude - log frequency curve never has a positive slope and that the phase shift is never greater than -90° for the two responses to valve area changes and the response to flow disturbance. For the surge valve area change, again the amplitude frequency response always has negative slope, but the phase angle reaches -180° at infinite frequency.

Two special cases which occur in systems of this kind will now be considered. In Appendix 1 it is shown that for pressure ratios across a restriction less than about 0.528, the flow is no longer dependent on downstream pressure; that is, in Appendix 1, $k_x = o$ and in the present problem, $k_3 = o$. This is the condition of a "choked" control valve in the present system, which results in a great simplification of the transfer function. Under this condition the transfer function consists of first order lags as shown in Appendix 2.

A second special case which is often encountered occurs when the constant k_e is infinite. One problem to be solved in gas flow systems is the need for a safety control which serves to keep the exhauster away from the surge region. This is accomplished by the recirculation surge valve shown in Appendix 2. Suppose this valve allows sufficient flow so that the exhauster is just at surge; that is, $k_e = \infty$. Then the system is again characterized by first order lags. (It should be noted that the problem of surge control can be attacked using the same governing laws.)

The Distributed Parameter Transfer Function

The distributed parameter description of plant response adds what the lumped parameter description fails to express--the spatial effects. Perhaps the most enlightening picture of the distributed system can be drawn from an examination of the physical interpretation of the solution to the partial differential equations. Equations (32) and (33) and more generally, equations (45) and (46) express this general solution. Allowing the Laplace operator in equation (46) to take on its general complex form, the term $(hc\,s^2 + rcs)^{1/2}$ can be expressed as the complex number $a + j\beta$. Then equation (46) can be written as

$$p\,(s) = E^*\, e^{ax}\, e^{j\beta x} + F^*\, e^{-ax}\, e^{-j\beta x}.$$

The first term of this expression represents a quantity that increases in magnitude and increases in phase as x increases. This term must, then, represent the contribution to the pressure change which originates at the point $x = L$, since it is logical to surmise that as a pressure disturbance travels toward $x = o$ from $x = L$ it experiences an attenuation and a time lag. By the same argument it can be concluded that the latter term in the expression, which decreases in magnitude and phase as x increases, must originate at the point, $x = o$. Thus, if a disturbance such as the flow disturbance previously mentioned originates at $x = o$, then all terms in the solution containing $e^{-ax}\, e^{-j\beta x}$ represent the original disturbance and successive reflections from $x = o$, whereas the terms containing $e^{ax}\; e^{j\beta x}$ must represent the successive reflections from the point $x = L$.

A number of special cases of the distributed parameter solutions may be treated, each of which adds to the understanding of a gas flow system. Most of these, however, are treated with other purposes in mind than control. Examples are the closed and open-end organ pipe, which in our interpretation represent problems of infinite and zero terminal impedance Z, the characterized line $(Z = \sqrt{\frac{h}{c}})$, and the infinite line $(e^{-ax} = o)$ as treated in electrical transmission-line studies and in the pneumatic instrument problem in which a long instrument line is terminated in an instrument volume. Most of these treatments consider only the resistanceless case such as was done in connection with the simple system of

Fig. 5. Space will not be given here to these special cases since they are amply treated in textbooks and reports.

Consideration of the resistance term has not been discussed. The effect of including the parameter can be understood most easily by numerical solution of the system represented in Fig. 5. This has been done in Appendix 3, where the calculated response curves are given. It is seen that the effect of the resistance is to smooth out the peaks and valleys in the frequency response. The numbers used in the calculations were chosen for ease of computation.

NATURE OF THE PARAMETERS

The properties of gas flow systems are the pneumatic capacitance, inertance and resistance terms (C, c, h, and r), and the perturbation constants--the partial derivatives evaluated at points denoted by k. These properties determine the system parameters, which are the gains, time constants, and propagation constants of the response equations. Considerable information can be gained about gas flow systems from a study of these properties and parameters.

Lumped Parameter High Frequency Approximation

In equation 27, for the region where rs (or $rj\omega$) is large compared to unity, then

$$\frac{p\,(j\omega)}{w_i\,(j\omega)} = \frac{1}{jC\omega} \quad .$$

The magnitude is $\frac{1}{C\omega}$ and the log magnitude vs log frequency plot is simply a straight line of slope, -1. Now, the property C ($\frac{V}{\gamma R\Theta}$) does not change with operating point (P_{ss} and W_{ss}), and thus the high frequency portion of the frequency response remains fixed. The system of Appendix 2 exhibits this same property also, where it can be shown that for high frequencies

$$\frac{p_i\ (j\omega)}{w_i\ (j\omega)} = \frac{1}{j\,C_1\,\omega}\ .$$

In fact, it can be shown that for all gas flow systems, using only the lumped parameter description, the high frequency region of the frequency response remains fixed and only the low frequency region changes with operating point.

For ease of plotting the frequency response, it is convenient to know the value of the point on the high frequency asymptote. The frequency at which the high frequency approximation log magnitude vs log frequency plot crosses unit magnitude is $\omega_1 = \frac{1}{C} = \frac{\gamma R \Theta}{V}$. Now, since Θ is absolute temperature, ω_1 does not shift appreciably with temperature. The volume is, then, the determining factor in the location of ω_1.

The Perturbation Constants

The use of the theory of small perturbations in system analysis is becoming widespread. Probably the largest field of use is in the aerodynamics of airframes where the partial derivatives, or perturbation constants when evaluated at a point, are known as stability derivatives. The evaluation of these stability derivatives of airframe dynamics is one of the principal projects of wind tunnel testing. The perturbation constants of the valves and compressors, for accurate analysis, must also be taken from experiment. However, one desires to be able to express them analytically. For a number of systems this can be done, as is shown for a control valve in Appendix 1.

The Distributed Parameters

Pneumatic resistance, capacitance, and inertance derive their names from their similarity to the equivalent electrical terms. The form of the resistance varies greatly with the system. For large ducts the only resistance may be due to the geometry of the system, such as tees, elbows, expansions or contractions. It is sometimes sufficient to consider the total resistance as being distributed evenly over the whole length. The inertance term $(\frac{1}{A_o g})$ becomes much larger for small diameter tubes than for large ducts, while for the capacitance term $(\frac{A_o g}{u^2})$ just the opposite is true.

Through the entire treatment the velocity of propagation term plays a major role. This is true also for the lumped pneumatic capacitance $(C = \frac{V}{\gamma R \Theta} = \frac{L A_o g}{u^2}$ and thus $C = Lc)$. The equivalence of the speed of sound in the pneumatic system and the speed of light in the electrical system should also be noted.

USE OF THE ANALYSIS IN CONTROL PROBLEMS

A number of factors influence the manner of analysis of gas flow systems. A brief cursory analysis of a system which already exists will reveal the basic properties. The magnitude of the largest volume and rough knowledge of temperature, pressure, and flow determine the largest and smallest time constants to be encountered. The shape of the system and its approximate length and reflection times quickly determine the critical frequencies; and, in fact, if one knows the design Mach number it becomes apparent immediately whether other than a lumped constant analysis need be considered.

In the design of new systems the requirements of the system determine what properties should be designed into it and which should be deleted for best performance. Thus, if it is required that the set point pressure in a system change rapidly, the time constant must be small and hence the volume small, whereas for systems in which it is required that pressure changes to disturbances be small, a large volume is dictated. In all cases it is desired that the volumes be short in length and large in cross section, thus eliminating distributed parameter effects as much as possible.

A detailed knowledge of the system transfer function with fairly accurate quantitative knowledge of the gains and time constants is necessary in control system design. If ω_b/ω_c is small, the lumped parameter expression is sufficient; if ω_b/ω_c is large, it is necessary at least to determine ω_c. Further information about the distributed parameter response is interesting academically and for understanding gas flow systems but does not add information helpful in control system design. It is sufficient to know that at a frequency equal to $2\omega_c$ the phase shift is already 180°, and hence a stability limit is reached. Present knowledge does not contribute a solution to this stability problem.

A convenient and informative method of presenting the variation of plant parameters is to "map" them on a pressure-weight flow plot. Consider, for example, the time constant in the simple system, $\tau = C/k_1$. This may be reduced to

$$P = \frac{\gamma R \Theta \tau}{V} W$$

which, for constant temperature, consists of a family of straight lines on the pressure-flow map. The constants of the system of Appendix 2 may be mapped similarly although the relations may become quite complicated at times.

With a knowledge of the order of the lumped parameter transfer function, the magnitude of the gains and time constants, and the distributed parameter critical frequencies, the control system designer can determine his controller components, the speed of response required of valves and sensing elements, and the limits of the control. Then, a fairly accurate prediction of errors and speeds of response to different forcings can be made.

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APPENDIX 1

COMPRESSIBLE FLOW THROUGH A CONTROL VALVE

The controlled area of a restriction, here referred to as a control valve, is nearly always the final control element in a gas flow system. The system designer must, then, have a fairly accurate knowledge of the behavior of the flow through restrictions. A qualitative knowledge (e.g., the flow is dependent on the pressure ratio, the area and temperature) is not sufficient; it is also necessary to know quantitatively how the flow is dependent on the variables.

A convenient manner of expressing this dependence is by a measure of the deviation of the actual, measured flow from a general, ideal relationship. That is

$$W_a = C_f \, W_t \, , \tag{68}$$

where C_f is a "flow coefficient" and subscripts a and t represent actual and theoretical flows, respectively.

The ideal flow relationship is derived in most texts on engineering thermodynamics (Ref. 3) from consideration of the continuity equation, the state and process equations of an ideal gas, and the general energy equation. A brief derivation of this ideal expression is given in this appendix.

The continuity equation can be expressed as

$$\rho_x A_x v_x = \rho_1 A_1 v_1 = \rho_2 A_2 v_2 = \text{-------} \; W \, , \tag{69}$$

where ρ is density, A is open area and v is velocity. The equation of state, $\rho = P/R\Theta$, and the expression for an adiabatic change of state, $(\rho_1/\rho_x) = (P_1/P_x)^{\frac{1}{\gamma}}$, are also of use. The general energy equation can be written, for the stations i and x as shown on Fig. 9, on a "per pound of fluid flowing" basis, as

$$P_1 V_1 - P_x V_x + JQ + \text{Work} = J(E_x - E_1) + \frac{v_x^2 - v_1^2}{2g} \, , \tag{70}$$

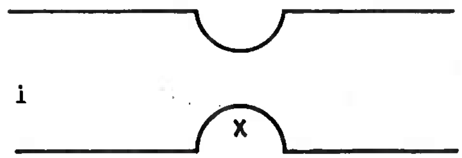

FIG. 9. SCHEMATIC OF RESTRICTION

where Q is heat added, E is internal energy and J is the heat-work conversion coefficient. If the process is adiabatic, then there is no net flow of heat or work. The internal energy is given by $C_v\,\Theta$, and by use of the equation of state and the adiabatic relationship the energy equation can be reduced to

$$\left(1 + \frac{JC_v}{R}\right) V_1 P_1 \left[1 - \left(\frac{P_x}{P_1}\right)^{\frac{\gamma-1}{\gamma}}\right] = \frac{v_x^2 - v_1^2}{2g} . \tag{71}$$

Remembering that $R = J C_v (\gamma - 1)$ equation (71) further reduces to

$$v_x = \left\{ \left(\frac{2\gamma R g}{\gamma - 1}\right) \Theta_1 \left[1 - \left(\frac{P_x}{P_1}\right)^{\frac{\gamma-1}{\gamma}}\right] + v_1^2 \right\}^{1/2} . \tag{72}$$

Now, using the continuity equation,

$$\frac{W}{A_x} = \rho_x v_x = \left\{ \left[\frac{2g\gamma}{R(\gamma-1)}\right] \frac{P_1^2}{\Theta_1} \left[\left(\frac{P_x}{P_1}\right)^{\frac{2}{\gamma}} - \left(\frac{P_x}{P_1}\right)^{\frac{\gamma+1}{\gamma}} \right] + \frac{v_1^2 P_1^2}{R^2 \Theta_1^2} \left(\frac{P_x}{P_1}\right)^{\frac{2}{\gamma}} \right\}^{\frac{1}{2}} , \tag{73}$$

further simplification yields

$$\frac{W}{A_x} = \left\{ \frac{2g\gamma}{R(\gamma-1)} \left(\frac{P_1}{\Theta_1}\right)^2 \frac{\left[\left(\frac{P_x}{P_1}\right)^{\frac{2}{\gamma}} - \left(\frac{P_x}{P_1}\right)^{\frac{\gamma+1}{\gamma}}\right]}{1 - \left(\frac{A_x}{A_1}\right)^2 \left(\frac{P_x}{P_1}\right)^{\frac{2}{\gamma}}} \right\}^{1/2} . \tag{74}$$

If A_1 is large compared to A_x, the denominator of expression (74) reduces to unity and the equation simplifies to

$$\frac{W}{A_x} = \sqrt{\frac{2g\gamma}{R(\gamma-1)}}\ \frac{P_1}{\sqrt{\Theta_1}} \left[\left(\frac{P_x}{P_1}\right)^{\frac{2}{\gamma}} - \left(\frac{P_x}{P_1}\right)^{\frac{\gamma+1}{\gamma}} \right]^{1/2} . \tag{75}$$

The function of the pressure ratio, $f\left(\frac{P_x}{P_1}\right)$, when plotted, exhibits a maximum which can be found by differentiation to occur at $\frac{P_x}{P_1} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$

$$(76)$$

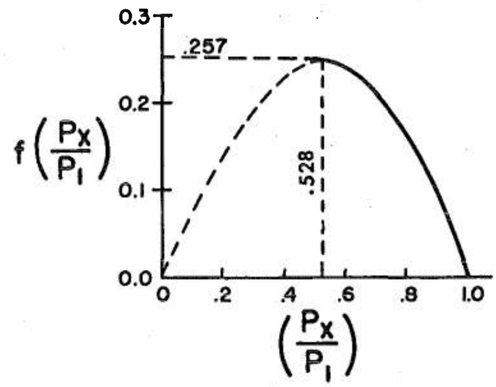

FIG. 10. PLOT OF FUNCTION OF PRESSURE RATIO

This value of the pressure ratio, when substituted in equation (72) yields

$$v_x = \sqrt{\frac{2\gamma g R \Theta_1}{\gamma + 1} + v_1^2} \tag{77}$$

and since $\Theta_1 = \Theta_x \left(\frac{P_1}{P_x}\right)^{\frac{\gamma-1}{\gamma}} = \Theta_x \left(\frac{\gamma+1}{2}\right)$ and $v_1 \ll v_x$ for $A_1 \gg A_x$, then

$$v_x = \sqrt{\gamma g R \Theta_x} \tag{78}$$

which is the sonic velocity. Once sonic velocity is achieved, no change in pressure occurring downstream of point x can be transmitted upstream and hence flow is independent of pressure ratios smaller than that indicated in equation (76). The flow for "choked" flows then reduces to

$$\frac{W}{A_x} = \frac{k \; P_1}{\sqrt{\Theta_1}} \; . \tag{79}$$

It is reasonable to suppose that the flow through a restriction will, after a fashion, obey the relationships (75) and (79). In fact, it seems that the flow relationship of an actual valve should differ from the theoretical expression by only a constant, the flow coefficient. That is, from equation (68),

$$W_a = C_f \; k \; \frac{P_1 A_x}{\sqrt{\Theta_1}} \; f\left(\frac{P_x}{P_1}\right) \tag{80}$$

where k and $f\left(\frac{P_x}{P_1}\right)$ have the obvious definitions.

To be valid, this approach to the problem should be checked by experiment for each type of control valve. This check has been made for a "butterfly" type control valve with very good agreement (Ref. 2).

Equation (80) may be written as

$$W_a = f(A_x, \Theta_1, P_x, P_1) .$$

Taking the partial differential of W_a we obtain

$$dW = \frac{\partial W}{\partial A_x} dA_x + \frac{\partial W}{\partial \Theta_1} d\Theta_1 + \frac{\partial W}{\partial P_x} dP_x + \frac{\partial W}{\partial P_1} dP_1 \tag{81}$$

or, by making certain definitions $w = k_a \, a + k_\theta \, \theta - k_x \, p_x + k_1 p_1$ (82)

where

$$a = dA_x \qquad k_\theta = \frac{\partial W}{\partial \Theta} = -\frac{1}{2}\frac{W}{\Theta}$$

$$\theta = d\Theta$$

$$p_x = dP_x \qquad k_x = -\frac{\partial W}{\partial P_x} = -\tfrac{1}{2}\frac{W}{P_1}\left(\frac{P_x}{P_1}\right)^{-1} \frac{\frac{2}{\gamma}\left(\frac{P_x}{P_1}\right)^{\frac{2}{\gamma}} - \frac{\gamma+1}{\gamma}\left(\frac{P_x}{P_1}\right)^{\frac{\gamma+1}{\gamma}}}{\left[\left(\frac{P_x}{P_1}\right)^{\frac{2}{\gamma}} - \left(\frac{P_x}{P_1}\right)^{\frac{\gamma+1}{\gamma}}\right]}$$

$$p_1 = dP_1$$

$$k_a = \frac{\partial W}{\partial A_x} = \frac{W}{A_x} \qquad k_1 = \frac{\partial W}{\partial P_1} = \frac{W}{P_1}\left(\frac{\gamma-1}{\gamma}\right) \frac{\left(\frac{P_x}{P_1}\right)^{\frac{2}{\gamma}} - \frac{1}{2}\left(\frac{P_x}{P_1}\right)^{\frac{\gamma+1}{\gamma}}}{\left(\frac{P_x}{P_1}\right)^{\frac{2}{\gamma}} - \left(\frac{P_x}{P_1}\right)^{\frac{\gamma+1}{\gamma}}}$$

The partial derivatives as expressed here suggest a convenient method of quantitative evaluation of the perturbation constants at an operating point, W *and* P_1. The quantities k_x and k_1 can be conveniently plotted in a dimensionless form as kP_1/W *vs.* P_x/P_1. It is also convenient to plot the ratio of weight flow to choked weight flow $\frac{W}{W_c}$ which is $\left[\frac{1}{f\left(\frac{P_x}{P_1}\right)}\right]_{\text{choked}} \left[f\left(\frac{P_x}{P_1}\right)\right] = 3.89 \; f\left(\frac{P_x}{P_1}\right)$. Thus, k_a which is $\frac{W}{A}$ can also be evaluated, since $\frac{k_a}{k_{a\,(\text{choked})}} = \frac{W}{W_c} = 3.89 \; f\left(\frac{P_x}{P_1}\right)$.

The functions k_a, k_1, and k_x are plotted in nondimensionalized form against P_x/P_1 on Figs. 11a, 11b and 11c, respectively. It is of interest to note that the flow coefficient is absent from these expressions.

It should be observed that the pressure P_x is the "throat" pressure and that the pressure downstream of the throat may be greater than P_x. This is considered as a "recovery" and introduces some error when throat and downstream pressure are considered equal.

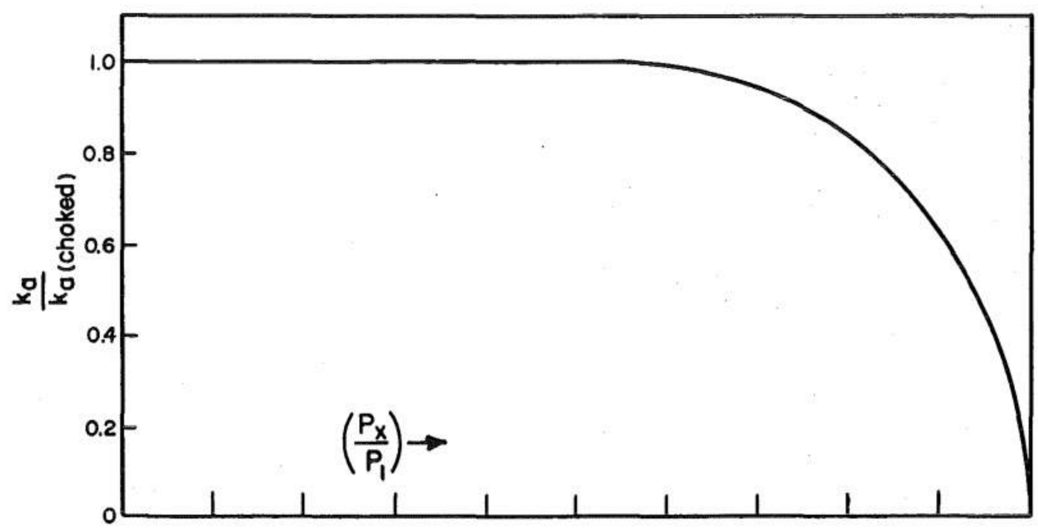

FIG. 11 (a). CHOCKED COEFFICIENT RATIO VS PRESSURE RATIO

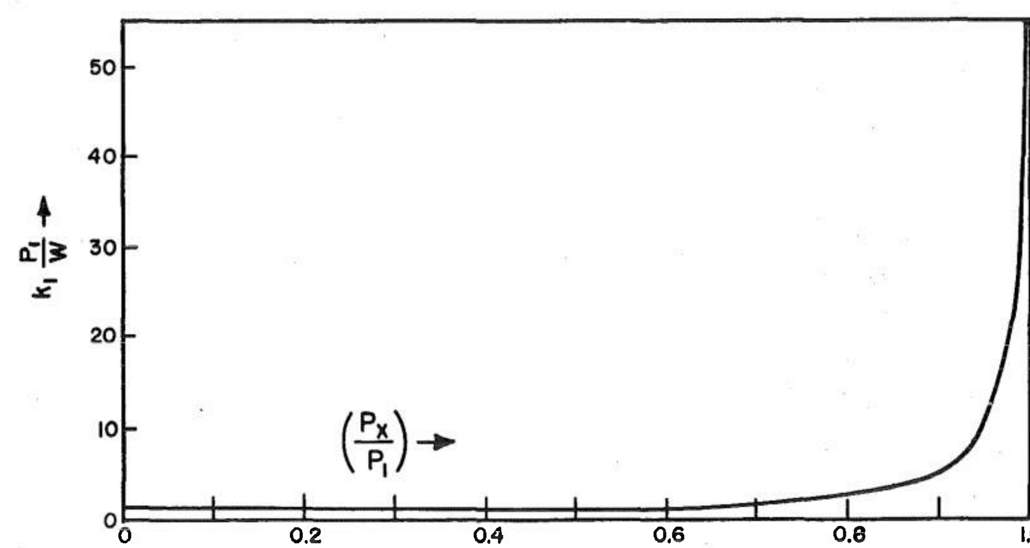

FIG. 11 (b). NON-DIMENSIONALIZED k_1 VS PRESSURE RATIO

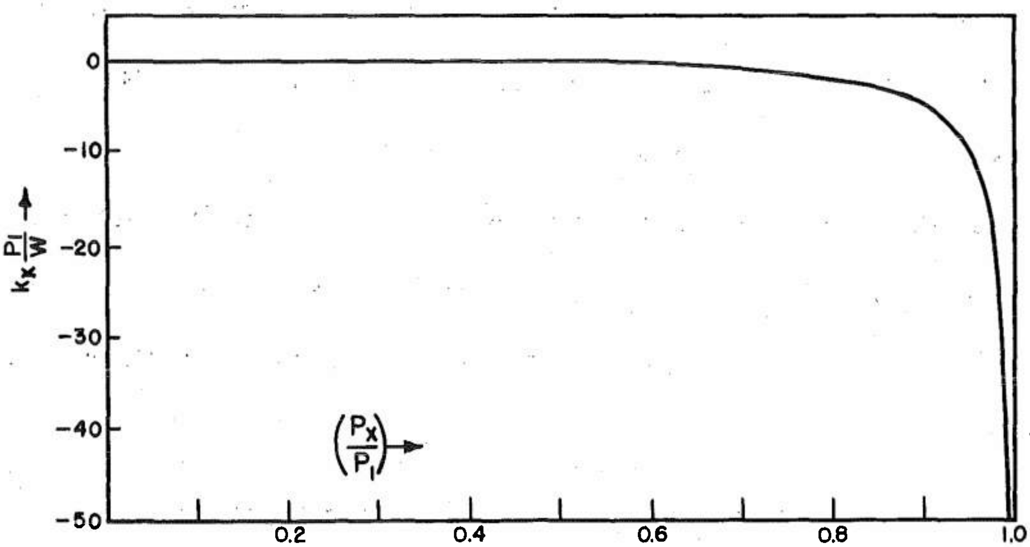

FIG. 11(c). NON-DIMENSIONALIZED k_x VS PRESSURE RATIO

APPENDIX 2

DETERMINATION OF THE TRANSFER FUNCTION OF A TYPICAL GAS FLOW SYSTEM

The system to be considered in this appendix may represent a number of different systems, but the author is familiar with it as the exhaust side of an engine-test plant. Referring to the symbolic diagram, Fig. 12, a quantity of gas W_i enters volume 1, which is some geometry of ducting. A quantity of gas W_b also enters volume 1 through the control bleed-in valve, the pressure P_1 being less than P_a. The total flow then enters volume 2 through the throttle control valve and is exhausted through a compressor to a pressure P_a. A part of the total flow W_s is recirculated around the exhauster and back into volume 2 for purposes of compressor surge control. The problem is to determine the dynamic relationship of the pressure in volume 1 as a function of an inlet flow disturbance and disturbances in the three valve areas. The methods of the lumped parameter analysis given in this paper will be used. Only the governing equations and the final transfer function will be given.

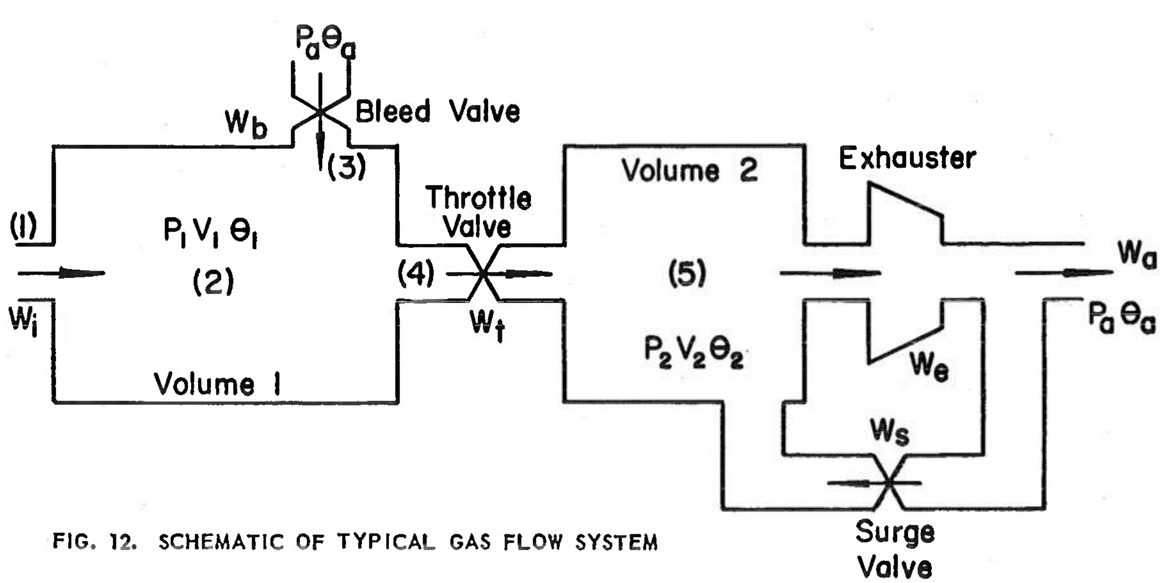

FIG. 12. SCHEMATIC OF TYPICAL GAS FLOW SYSTEM

Conditions: $\Theta_1 = \Theta_2$

P_a and Θ_a are constant.

Governing Equations:

$$P_1 = \frac{1}{C_i s}(W_i + W_b - W_t); \quad p_1 = \frac{1}{C_i s}(w_i + w_b - w_t) \tag{84}$$

$$W_b = \frac{KP_aA_b}{\sqrt{\Theta_a}} f\left(\frac{P_1}{P_a}\right); \quad w_b = -k_1 p_1 + k_b a_b \tag{85}$$

$$W_t = \frac{KP_1A_t}{\sqrt{\Theta_1}} f\left(\frac{P_2}{P_1}\right); \quad w_t = k_2 p_1 + k_t a_t - k_3 p_2 \tag{86}$$

$$P_2 = \frac{1}{C_2 s}(W_t + W_s - W_e); \quad p_2 = \frac{1}{C_2 s}(w_t + w_s - w_e) \tag{87}$$

$$W_e = f(P_2); \quad w_e = k_e p_2 \tag{88}$$

$$W_s = \frac{KP_aA_s}{\sqrt{\Theta_a}} f\left(\frac{P_2}{P_a}\right); \quad w_s = k_s a_s - k_5 p_2 \tag{89}$$

Transfer Function: Response of P_1 to changes in W_i, A_b, A_t, A_s is:

$$p_1(s) = \frac{K_i(r_i s + 1)}{\frac{1}{\omega_n^2}s^2 + \frac{2}{\omega_n}\zeta s + 1} w_i + \frac{K_b(r_1 s + 1)}{\frac{1}{\omega_n^2}s^2 + \frac{2}{\omega_n}\zeta s + 1} a_b - \frac{K_t(r_2 s + 1)}{\frac{1}{\omega_n^2}s^2 + \frac{2}{\omega_n}\zeta s + 1} a_t$$

$$+ \frac{K_s}{\frac{1}{\omega_n^2}s^2 + \frac{2}{\omega_n}\zeta s + 1} a_s;$$

or,

$$p_1(s) = \frac{K_i(r_1 s + 1)\,w}{(r_3 s + 1)(r_4 s + 1)} w_i + \frac{K_b(r_1 s + 1)}{(r_3 s + 1)(r_4 s + 1)} a_b - \frac{K_t(r_2 s + 1)}{(r_3 s + 1)(r_4 s + 1)} a_t + \frac{K_s}{(r_3 s + 1)(r_4 s + 1)} a_s$$

Parameters:

$$K_i = \frac{k_3 + k_5 + k_e}{k_1 k_3 + k_1 k_5 + k_1 k_e + k_2 k_5 + k_2 k_e}$$

$$K_b = \frac{k_b \, (k_3 + k_5 + k_e)}{k_1 k_3 + k_1 k_5 + k_1 k_e + k_2 k_5 + k_2 k_e}$$

$$K_t = \frac{k_t \, (k_5 + k_e)}{k_1 k_3 + k_1 k_5 + k_1 k_e + k_2 k_5 + k_2 k_e}$$

$$K_s = \frac{k_3 k_5}{k_1 k_3 + k_1 k_5 + k_1 k_e + k_2 k_5 + k_2 k_e}$$

$$w_n = \frac{1}{\sqrt{C_1 C_2}} \quad \sqrt{k_1 k_3 + k_1 k_5 + k_1 k_e + k_2 k_5 + k_2 k_e}$$

$$\zeta = \frac{1}{2} \; \frac{\sqrt{\frac{C_2}{C_1}} \, (k_3 + k_5 + k_e) + \sqrt{\frac{C_1}{C_2}} \, (k_1 + k_2)}{\sqrt{k_1 k_3 + k_1 k_5 + k_1 k_e + k_2 k_5 + k_2 k_e}}$$

$$\tau_1 = \frac{C_2}{k_3 + k_5 + k_e} \; ; \quad \tau_2 = \frac{C_2}{k_5 + k_e}$$

$$\tau_3 , \tau_4 = \frac{1}{\frac{1}{2}\left[\frac{k_3 + k_5 + k_e}{C_2} + \frac{k_1 + k_2}{C_1}\right] \pm \frac{1}{2}\left[\left(\frac{k_3 + k_5 + k_e}{C_2} - \frac{k_1 + k_2}{C_1}\right)^2 + \frac{4 \, k_2 k_3}{C_1 C_2}\right]^{1/2}}$$

Special Case of a Choked Control Valve:

$$k_3 = 0$$

$$\tau_3 = \tau_2 = \tau_1 \; ; \quad \tau_4 = \frac{C_1}{k_1 + k_2}$$

$$K_i = \frac{1}{k_1 + k_2} \; ; \quad K_b = \frac{k_b}{k_1 + k_2} \; , \quad K_t = \frac{k_t}{k_1 + k_2} \; , \quad K_s = 0$$

$$\frac{p_1(s)}{w_i(s)} = \frac{K_i}{\tau_4 s + 1} \, w_i + \frac{K_b}{\tau_4 s + 1} \, a_b - \frac{K_t}{\tau_4 s + 1} \, a_t$$

Special Case of Exhauster at Surge Flow:

$$k_e \doteq \infty$$

$$\tau_1 = \tau_2 = 0; \ \frac{1}{\omega_n^2} = 0, \ \frac{2\zeta}{\omega_n} = \frac{C_1}{k_1 + k_2} = \tau_5$$

$$K_i = \frac{1}{k_1 + k_2}, \quad K_b = \frac{k_b}{k_1 + k_2}, \quad K_t = \frac{k_t}{k_1 + k_2}$$

$$\frac{p_1(s)}{w_i(s)} = \frac{K_i}{\tau_5 s + 1} w_i + \frac{K_b}{\tau_5 s + 1} a_b - \frac{K_t}{\tau_5 s + 1} a_t$$

APPENDIX 3

NUMERICAL CALCULATION OF THE RESPONSE OF A SIMPLE SYSTEM

In order to show more definitely the response of a system and the effect of different properties of the system, the response of the system of Fig. 13 is computed. The numbers have been chosen so as to minimize the computations. The ft-lb-sec unit system is used.

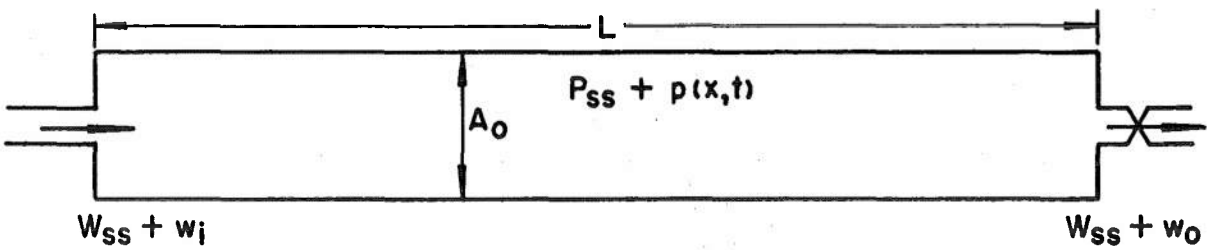

FIG. 13. SCHEMATIC OF A SIMPLE SYSTEM

The properties and parameters chosen for the system are as follows:

Properties

Duct length, $L = 50\pi$ ft

Duct cross-sectional area, $A_o = \dfrac{2000}{\pi g}$ ft^2

Steady state pressure, $P_{ss} = 1000$ lbs/ft^2

Steady state flow, $W_{ss} = 100$ lbs/sec

Steady state temperature, $\Theta_{ss} = 416°R$

Velocity of wave propagation, $u = 1000$ ft/sec

Lumped pneumatic capacitance, $C = \dfrac{V}{\gamma R\Theta} = 0.1$ ft^2

Distributed pneumatic capacitance, $c = \dfrac{gA_o}{u^2} = \dfrac{2}{\pi} \times 10^{-3}$ ft

Distributed pneumatic inertance, $h = \dfrac{1}{A_o g} = \dfrac{\pi}{2} \times 10^{-3} \dfrac{\text{sec}^2}{\text{ft}^3}$

Distributed pneumatic resistance, $r = \dfrac{\Delta P}{W_{ss}} = \dfrac{\pi}{2} \times 10^{-3} \dfrac{\text{sec}}{\text{ft}^3}$

Parameters

Terminal pneumatic impedance, $Z = \dfrac{1}{K_o} = 10 \text{ sec/ft}^2$

Lumped parameter time constant, $r = \dfrac{C}{K_o} = 1 \text{ sec}$

Distributed parameter critical frequency, $\omega_c = \dfrac{\pi u}{2L} = 10 \text{ rad/sec}$

Lumped Parameter Frequency Response

The lumped parameter transfer function can be written numerically as

$$\frac{p(s)}{w_i(s)} = \frac{1}{k_o(rs+1)} = \frac{10}{s+1} \tag{90}$$

or replacing s by $j\omega$, the frequency response is

$$\frac{p(j\omega)}{w_i(j\omega)} = \frac{10}{j\omega + 1}\,. \tag{91}$$

The frequency response is shown on Figs. 14 and 15 for the log-log plot of magnitude ratio vs frequency and the semi-log plot of phase angle vs frequency, respectively.

Distributed Parameter Frequency Response

The distributed parameter frequency response transfer function can be written as

$$\frac{p(x, j\omega)}{w_i(x, j\omega)} = \frac{f_1 + jf_2}{f_3 + jf_4}\,, \tag{92}$$

where the functions f are given by

$$f_1 = Z\cos(\beta x)\cosh(ax) + \frac{\beta}{\omega c}\cos(\beta x)\sinh(ax) + \frac{a}{\omega c}\sin(\beta x)\cosh(ax)$$

$$f_2 = Z\sin(\beta x)\sinh(ax) + \frac{\beta}{\omega c}\sin(\beta x)\cosh(ax) - \frac{a}{\omega c}\cos(\beta x)\sinh(ax)$$

$$f_3 = \cos(\beta L)\cosh(aL) + \frac{Z\,\omega c\beta}{a^2+\beta^2}\cos(\beta L)\sinh(aL) - \frac{Z\,\omega ca}{a^2+\beta^2}\sin(\beta L)\cosh(aL)$$

$$f_4 = \sin(\beta L)\sinh(aL) + \frac{Z\,\omega c\beta}{a^2+\beta^2}\sin(\beta L)\cosh(aL) + \frac{Z\,\omega ca}{a^2+\beta^2}\cos(\beta L)\sinh(aL)\,.$$

The numbers a and β are given by

$$\beta = \sqrt{\frac{1}{2} \omega c \left[\omega h + \sqrt{\omega^2 h^2 + r^2} \right]}$$

$$a = \frac{\omega r c}{2\beta}$$

Case 1: $r = 0$

Under this special case the functions simplify greatly; $\beta = \frac{\omega}{u}$ and $a = 0$, and the functions f become

$$f_1 = Z \cos \beta x = 10 \cos \frac{\omega x}{1000}$$

$$f_2 = \frac{\beta}{\omega c} \sin \beta x = \frac{\pi}{2} \sin \frac{\omega x}{1000}$$

$$f_3 = \cos \beta L = \cos 0.05\, \pi\omega$$

$$f_4 = \frac{Z\, \omega c}{\beta} \sin \beta L = \frac{20}{\pi} \sin 0.05\, \pi\omega\,.$$

Two positions of x are of interest, $x = 0$ and $x = L$. Under these conditions the response curves are shown in Figs. 14 and 15.

Case 2: $r \neq 0$

For this more general case the response is more difficult to calculate. Our choice of r makes the calculations simpler however. Since r was chosen such that $r = h$, the expressions reduce to

$$\beta = \frac{1}{u} \sqrt{\frac{1}{2} \omega \left(\omega + \sqrt{\omega^2 + 1}\right)}$$

$$a = \frac{1}{2u} \sqrt{\frac{2\omega}{\omega + \sqrt{\omega^2 + 1}}}$$

Point by point calculation of the response then yields the curves shown in Figs. 14 and 15.

Lumped Parameter Indicial Response

The solution of the transfer function (90) for $w_i\ (s) = \frac{w_i}{s}$, a unit step function, is

$$p(t) = \frac{|w_i|}{k_o} \left(1 - e^{-\frac{t}{\tau}}\right) = 10\left(1 - e^{-t}\right). \qquad (93)$$

This response is plotted as a function of time on Fig. 16.

Distributed Parameter Indicial Response

The indicial response for the distributed, resistanceless system, is given by the expression

$$p(x,\, t) = Z\,|w_i|\left\{1 - \frac{1}{2}(1 - \delta)\left[1 + \left(\frac{1+\delta}{1-\delta}\right)^{\frac{x}{L}}\right]\left[\frac{1-\delta}{1+\delta}\right]^{n}\right\} \tag{94}$$

$$(2n\,L - x) < \frac{t}{\sqrt{hc}} < 2(n + 1)\,L - x \qquad n = 0,\, 1,\, 2,\, \cdots,$$

where δ is defined by the ratio $\dfrac{\sqrt{\frac{h}{c}}}{Z_t}$. For the two positions of interest, $x = 0$ and $x = L$, the responses for the unit step input are

at $x = 0$

$$\frac{p(0,\, t)}{Z_t} = 1 - \frac{(1 - \delta)^{n+1}}{(1 + \delta)^{n}} = 1 - \frac{(20 - \pi)^{n+1}}{(20 + \pi)^{n}} \tag{95}$$

$$2n < \frac{t}{.05\pi} < 2\,(n + 1)$$

at $x = L$

$$\frac{p(L,\, t)}{Z_t} = 1 - \left(\frac{1 - \delta}{1 + \delta}\right)^{n} = 1 - \left(\frac{20 - \pi}{20 + \pi}\right)^{n} \tag{96}$$

$$(2n - 1) < \frac{t}{.05\pi} < (2n + 1)$$

This plot also appears on Fig. 16

Conclusions

From these calculated responses one gets an insight into the nature of gas flow systems. The effect of resistance on the frequency response should be particularly noted. It would be well also to have the indicial response for the case of a finite resistance but this calculation would be very difficult.

The following conclusions and observations can be made on the frequency response of this system:

1. The departure of the distributed parameter response from the lumped

parameter response becomes appreciable at about half way between the lumped parameter break frequency and the frequency at which the first quarter wave length occurs.

2. The effect of resistance on the distributed parameter response is to decrease the magnitude and round off the peaks of the periodic maxima and minima.

3. The impedance (equal to the magnitude ratio $\frac{p}{w_i}$, which is seen at the inlet of the system) is increased by an amount equal to the total line resistance (rL). The outlet impedance does not experience this effect. This is shown by the positions on the curves on Fig. 14 at $\omega = 0.1$ radians per second.

The following conclusions and observations can be made on the indicial response:

1. At the outlet, no effect of a disturbance at the inlet is felt until a time 0.05π sec, which is the time required for the transmission of a pressure wave. Thereafter steps are 0.1π sec apart, the time required for a pressure wave to traverse the length in both directions.

2. The pattern of the successive steps shows remarkable resemblance to the first order response.

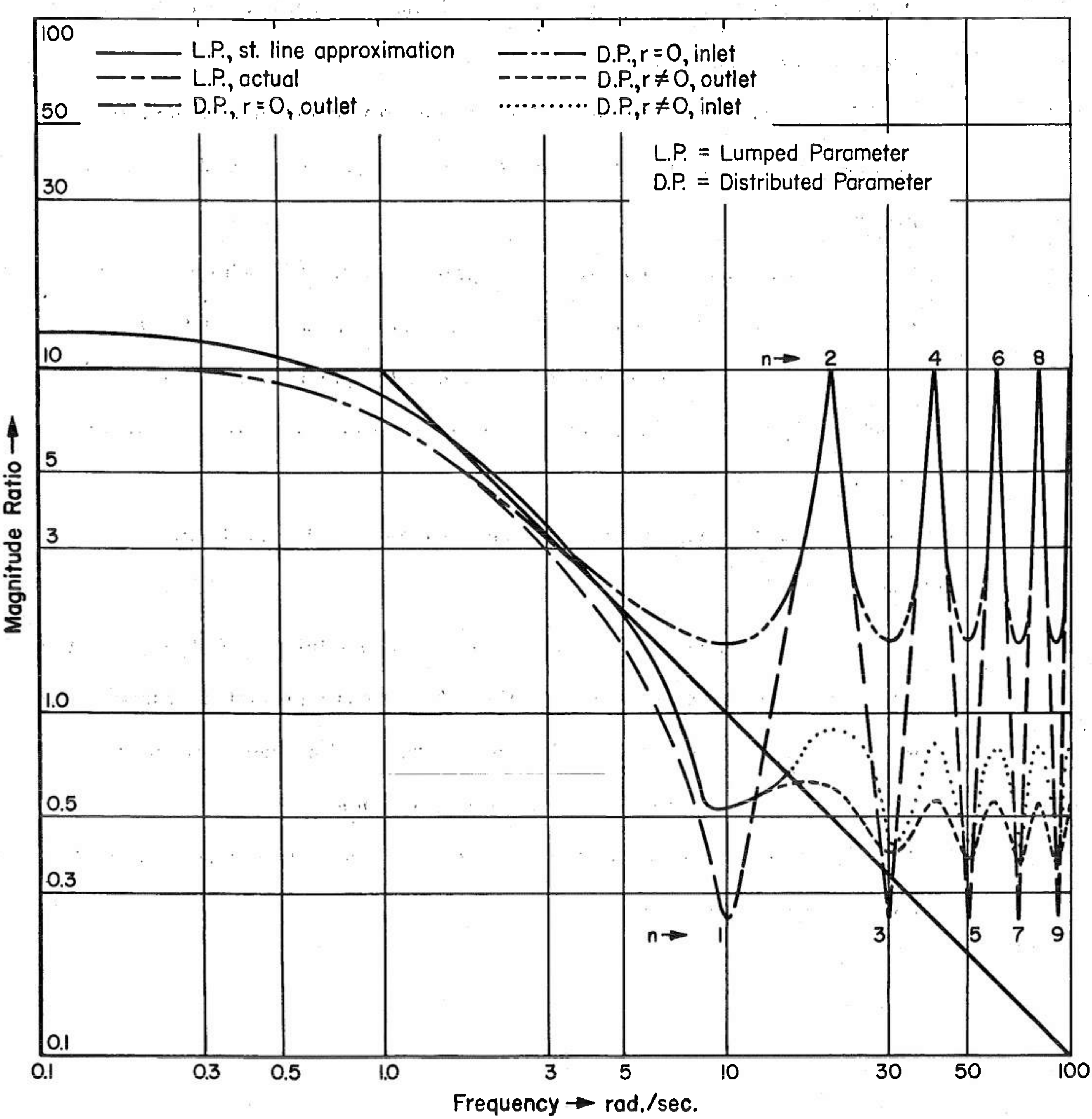

FIG. 14. FREQUENCY RESPONSE -- MAGNITUDE RATIO

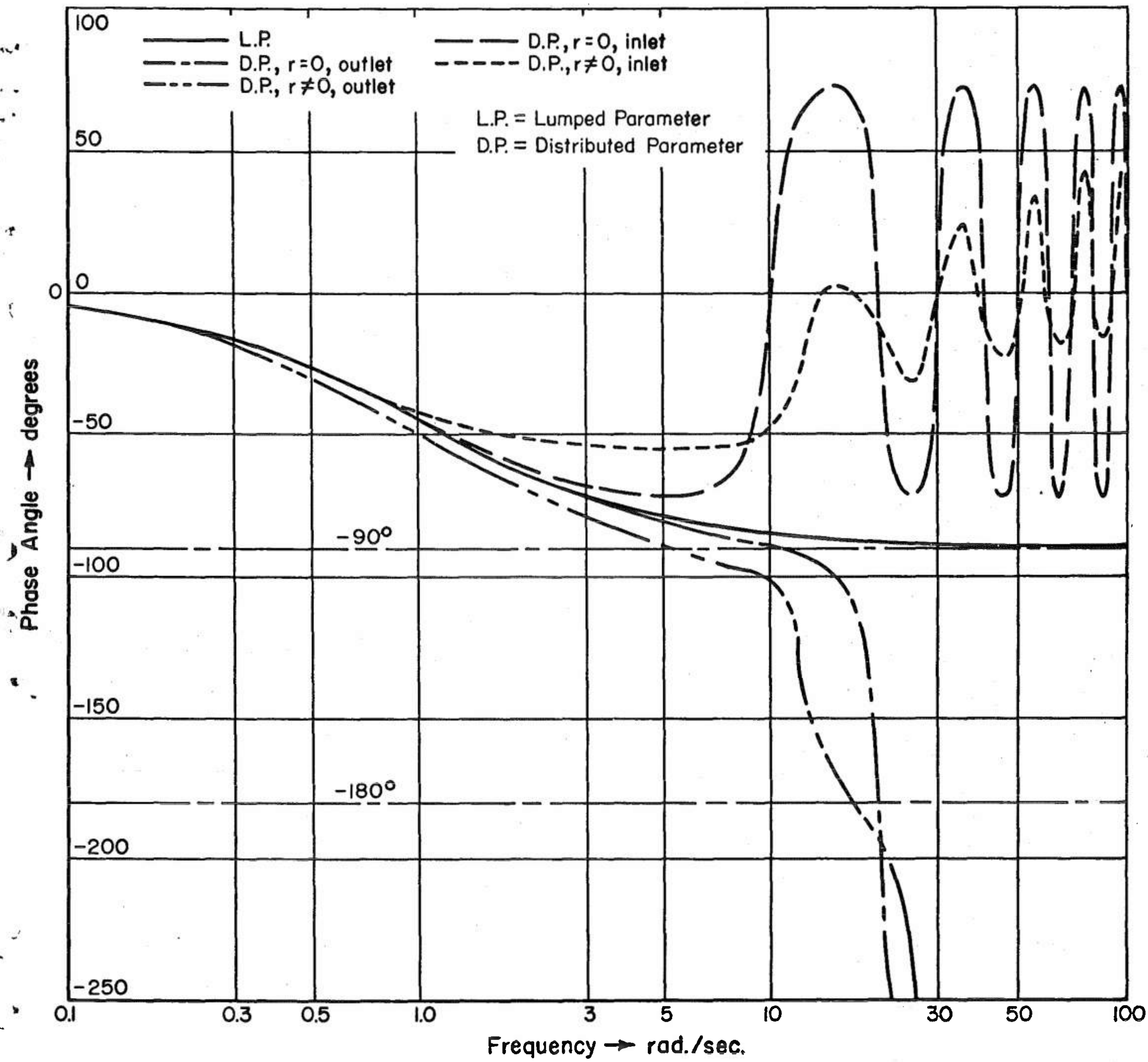

FIG. 15. FREQUENCY RESPONSE -- PHASE ANGLE

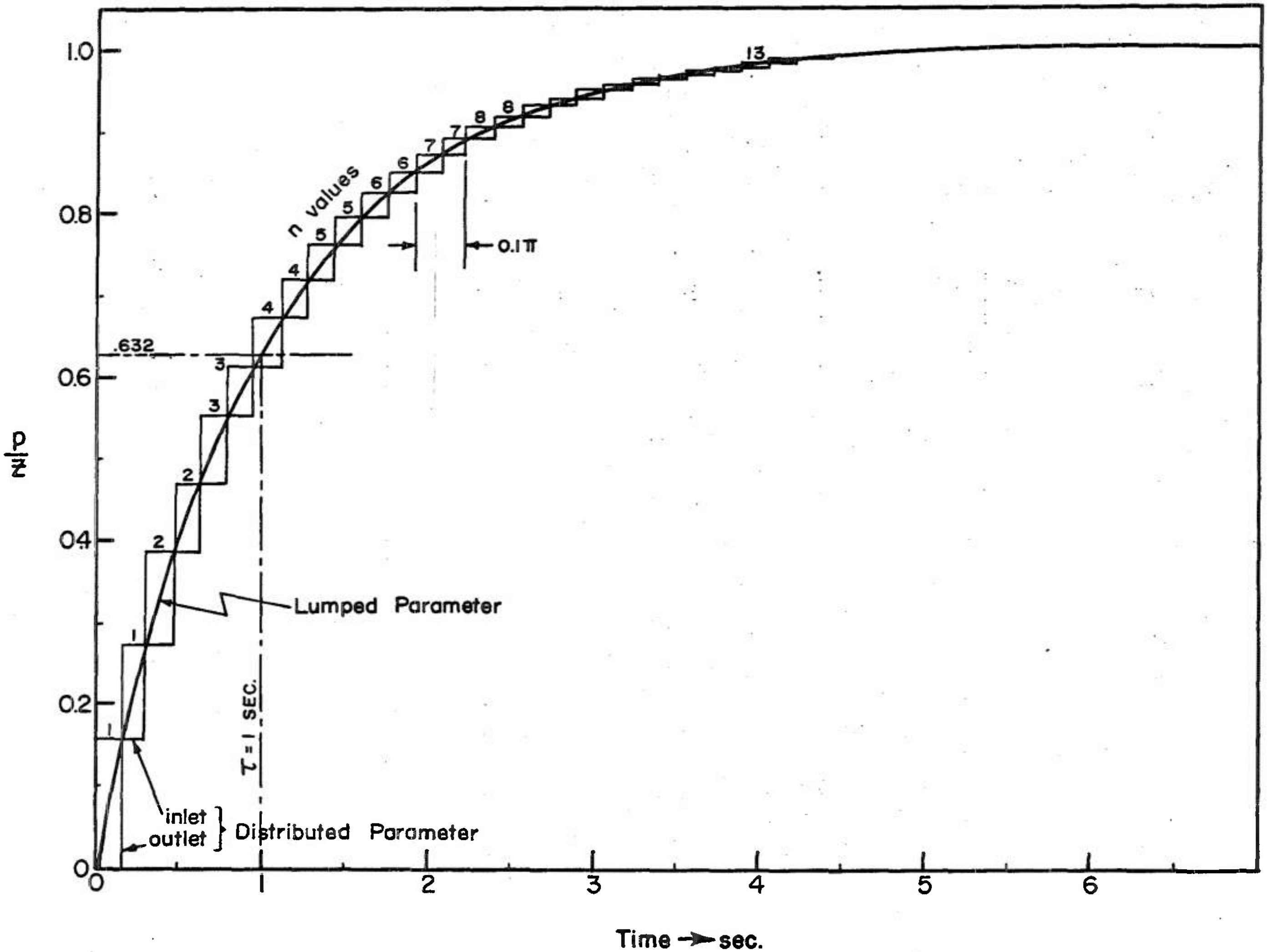

FIG. 16. INDICIAL RESPONSE